7

Method of Separation of Variables

"However, the emphasis should be somewhat more on how to do the mathematics quickly and easily, and what formulas are true, rather than the mathematicians' interest in methods of rigorous proof."

Richard Feynman

"As a *science*, mathematics has been adapted to the description of natural phenomena, and the great practitioners in this field, such as von Kármán, Taylor and Lighthill, have never concerned themselves with the logical foundations of mathematics, but have boldly taken a pragmatic view of mathematics as an intellectual machine which works successfully. Description has been verified by further observation, still more strikingly by prediction, ... "

George Temple

7.1 Introduction

The method of separation of variables combined with the principle of superposition is widely used to solve initial boundary-value problems involving linear partial differential equations. Usually, the dependent variable $u(x, y)$ is expressed in the separable form $u(x, y) = X(x)\,Y(y)$, where X and Y are functions of x and y respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. A similar treatment can be applied to equations in three or more independent variables. However, the question of separability of a partial differential equation into two or more ordinary differential equations is by no means a trivial one. In spite of this question, the method is widely used in finding solutions of a large class of initial boundary-value problems. This method of solution is also known as the *Fourier method* (or *the method of eigenfunction expansion*). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions, and orthogonality, all of which are very general and powerful for dealing with linear problems. The following examples illustrate the general nature of this method of solution.

7.2 Separation of Variables

In this section, we shall introduce one of the most common and elementary methods, called the *method of separation of variables*, for solving initial boundary-value problems. The class of problems for which this method is applicable contains a wide range of problems of mathematical physics, applied mathematics, and engineering science.

We now describe the method of separation of variables and examine the conditions of applicability of the method to problems which involve second-order partial differential equations in two independent variables.

We consider the second-order homogeneous partial differential equation

$$a^* u_{x^* x^*} + b^* u_{x^* y^*} + c^* u_{y^* y^*} + d^* u_{x^*} + e^* u_{y^*} + f^* u = 0 \qquad (7.2.1)$$

where a^*, b^*, c^*, d^*, e^* and f^* are functions of x^* and y^*.

We have stated in Chapter 4 that by the transformation

$$x = x(x^*, y^*), \qquad y = y(x^*, y^*), \qquad (7.2.2)$$

where

$$\frac{\partial(x, y)}{\partial(x^*, y^*)} \neq 0,$$

we can always transform equation (7.2.1) into canonical form

$$a(x, y) u_{xx} + c(x, y) u_{yy} + d(x, y) u_x + e(x, y) u_y + f(x, y) u = 0, \qquad (7.2.3)$$

which when

(i) $a = -c$ is hyperbolic,
(ii) $a = 0$ or $c = 0$ is parabolic,
(iii) $a = c$ is elliptic.

We assume a separable solution of (7.2.3) in the form

$$u(x, y) = X(x) Y(y) \neq 0, \qquad (7.2.4)$$

where X and Y are, respectively, functions of x and of y alone, and are twice continuously differentiable. Substituting equations (7.2.4) into equation (7.2.3), we obtain

$$a\,X''Y + c\,XY'' + d\,X'Y + e\,XY' + f\,XY = 0, \tag{7.2.5}$$

where the primes denote differentiation with respect to the appropriate variables. Let there exist a function $p(x, y)$, such that, if we divide equation (7.2.5) by $p(x, y)$, we obtain

$$a_1(x)\,X''Y + b_1(y)\,XY'' + a_2(x)\,X'Y + b_2(y)\,XY' + [a_3(x) + b_3(y)]\,XY = 0. \tag{7.2.6}$$

Dividing equation (7.2.6) again by XY, we obtain

f(X",X',X,x)=f(Y",Y',Y,y)

$$\left[a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3\right] = -\left[b_1\frac{Y''}{Y} + b_2\frac{Y'}{Y} + b_3\right]. \tag{7.2.7}$$

The left side of equation (7.2.7) is a function of x only. The right side of equation (7.2.7) depends only upon y. Thus, we differentiate equation (7.2.7) with respect to x to obtain

$$\frac{d}{dx}\left[a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3\right] = 0. \tag{7.2.8}$$

Integration of equation (7.2.8) yields

Solve This ODE

$$a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3 = \lambda, \tag{7.2.9}$$

where λ is a separation constant. From equations (7.2.7) and (7.2.9), we have

Solve this ODE too

$$b_1\frac{Y''}{Y} + b_2\frac{Y'}{Y} + b_3 = -\lambda. \tag{7.2.10}$$

We may rewrite equations (7.2.9) and (7.2.10) in the form

$$a_1X'' + a_2X' + (a_3 - \lambda)\,X = 0, \tag{7.2.11}$$

and

$$b_1Y'' + b_2Y' + (b_3 + \lambda)\,Y = 0. \tag{7.2.12}$$

Thus, $u(x, y)$ is the solution of equation (7.2.3) if $X(x)$ and $Y(y)$ are the solutions of the ordinary differential equations (7.2.11) and (7.2.12) respectively.

If the coefficients in equation (7.2.1) are constant, then the reduction of equation (7.2.1) to canonical form is no longer necessary. To illustrate this, we consider the second-order equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0, \tag{7.2.13}$$

where A, B, C, D, E, and F are constants which are not all zero.

As before, we assume a separable solution in the form

$$u(x,y) = X(x)\,Y(y) \neq 0.$$

Substituting this in equation (7.2.13), we obtain

$$AX''Y + BX'Y' + CXY'' + DX'Y + EXY' + FXY = 0. \tag{7.2.14}$$

Division of this equation by AXY yields

$$\frac{X''}{X} + \frac{B}{A}\frac{X'}{X}\frac{Y'}{Y} + \frac{C}{A}\frac{Y''}{Y} + \frac{D}{A}\frac{X'}{X} + \frac{E}{A}\frac{Y'}{Y} + \frac{F}{A} = 0, \quad A \neq 0. \tag{7.2.15}$$

We differentiate this equation with respect to x to obtain

$$\left(\frac{X''}{X}\right)' + \frac{B}{A}\left(\frac{X'}{X}\right)'\frac{Y'}{Y} + \frac{D}{A}\left(\frac{X'}{X}\right)' = 0. \tag{7.2.16}$$

Thus, we have

$$\frac{\left(\frac{X''}{X}\right)'}{\frac{B}{A}\left(\frac{X'}{X}\right)'} + \frac{D}{B} = -\frac{Y'}{Y}. \tag{7.2.17}$$

This equation is obviously separable, so that both sides must be equal to a constant λ. Therefore, we obtain

$$Y' + \lambda Y = 0, \tag{7.2.18}$$

$$\left(\frac{X''}{X}\right)' + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{X'}{X}\right)' = 0. \tag{7.2.19}$$

Integrating equation (7.2.19) with respect to x, we obtain

$$\frac{X''}{X} + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{X'}{X}\right) = -\beta, \tag{7.2.20}$$

where β is a constant to be determined. Substituting equation (7.2.18) into the original equation (7.2.15), we obtain

$$X'' + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}X' + \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}X = 0. \tag{7.2.21}$$

Comparing equations (7.2.20) and (7.2.21), we clearly find

$$\beta = \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}.$$

Therefore, $u(x,y)$ is a solution of equations (7.2.13) if $X(x)$ and $Y(y)$ satisfy the ordinary differential equations (7.2.21) and (7.2.18) respectively.

We have just described the conditions on the separability of a given partial differential equation. Now, we shall take a look at the boundary conditions involved. There are several types of boundary conditions. The ones that appear most frequently in problems of applied mathematics and mathematical physics include

(i) Dirichlet condition: u is prescribed on a boundary
(ii) Neumann condition: $(\partial u/\partial n)$ is prescribed on a boundary
(iii) Mixed condition: $(\partial u/\partial n) + hu$ is prescribed on a boundary, where $(\partial u/\partial n)$ is the directional derivative of u along the outward normal to the boundary, and h is a given continuous function on the boundary. For details, see Chapter 9 on boundary-value problems.

Besides these three boundary conditions, also known as, the *first*, *second*, and *third boundary conditions*, there are other conditions, such as the Robin condition; one condition is prescribed on one portion of a boundary and another is given on the remainder of the boundary. We shall consider a variety of boundary conditions as we treat problems later.

To separate boundary conditions, such as the ones listed above, it is best to choose a coordinate system suitable to a boundary. For instance, we choose the Cartesian coordinate system (x, y) for a rectangular region such that the boundary is described by the coordinate lines $x =$ constant and $y =$ constant, and the polar coordinate system (r, θ) for a circular region so that the boundary is described by the lines $r =$ constant and $\theta =$ constant.

Another condition that must be imposed on the separability of boundary conditions is that boundary conditions, say at $x = x_0$, must contain the derivatives of u with respect to x only, and their coefficients must depend only on x. For example, the boundary condition

$$[u + u_y]_{x=x_0} = 0$$

cannot be separated. Needless to say, a mixed condition, such as $u_x + u_y$, cannot be prescribed on an axis.

7.3 The Vibrating String Problem

As a first example, we shall consider the problem of a vibrating string of constant tension T^* and density ρ with $c^2 = T^*/\rho$ stretched along the x-axis from 0 to l, fixed at its end points. We have seen in Chapter 5 that the problem is given by

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < l, \qquad t > 0, \tag{7.3.1}$$

$$u(x, 0) = f(x), \qquad 0 \le x \le l, \tag{7.3.2}$$

$$u_t(x, 0) = g(x), \qquad 0 \le x \le l, \tag{7.3.3}$$

$$u(0, t) = 0, \qquad t \ge 0, \tag{7.3.4}$$

$$u(l, t) = 0, \qquad t \ge 0, \tag{7.3.5}$$

where f and g are the initial displacement and initial velocity respectively.

By the method of separation of variables, we assume a solution in the form

$$u(x,t) = X(x)\,T(t) \neq 0. \tag{7.3.6}$$

If we substitute equation (7.3.6) into equation (7.3.1), we obtain

$$XT'' = c^2 X'' T,$$

and hence,

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}, \tag{7.3.7}$$

whenever $XT \neq 0$. Since the left side of equation (7.3.7) is independent of t and the right side is independent of x, we must have

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda,$$

f(X",X,x)=f(T",T,t)=parameter

where λ is a separation constant. Thus,

$$X'' - \lambda X = 0, \tag{7.3.8}$$

$$T'' - \lambda c^2 T = 0. \tag{7.3.9}$$

We now separate the boundary conditions. From equations (7.3.4) and (7.3.6), we obtain

$$u(0,t) = X(0)\,T(t) = 0.$$

We know that $T(t) \neq 0$ for all values of t, therefore,

$$X(0) = 0. \tag{7.3.10}$$

In a similar manner, boundary condition (7.3.5) implies

$$X(l) = 0. \tag{7.3.11}$$

To determine $X(x)$ we first solve the *eigenvalue problem* (eigenvalue problems are also treated in Chapter 8)

$$X'' - \lambda X = 0, \qquad X(0) = 0, \qquad X(l) = 0. \tag{7.3.12}$$

We look for values of λ which gives us nontrivial solutions. We consider three possible cases

$$\lambda > 0, \qquad \lambda = 0, \qquad \lambda < 0.$$

Case 1. $\lambda > 0$. The general solution in this case is of the form

$$X(x) = Ae^{-\sqrt{\lambda}\,x} + Be^{\sqrt{\lambda}\,x}$$

where A and B are arbitrary constants. To satisfy the boundary conditions, we must have

A+B=0
Ae^(-lm)+Be^(lm)=0

$$A + B = 0, \qquad Ae^{-\sqrt{\lambda}\,l} + Be^{\sqrt{\lambda}\,l} = 0. \tag{7.3.13}$$

We see that the determinant of the system (7.3.13) is different from zero. Consequently, A and B must both be zero, and hence, the general solution $X(x)$ is identically zero. The solution is trivial and hence, is no interest.

Case 2. $\lambda = 0$. Here, the general solution is

Since determinant of coefficient matrix is non zero

sol of System of linear equations is zero i.e. A=B=0

$$X(x) = A + Bx.$$

Applying the boundary conditions, we have

$$A = 0, \qquad A + Bl = 0.$$

Hence $A = B = 0$. The solution is thus identically zero.

Case 3. $\lambda < 0$. In this case, the general solution assumes the form

$$X(x) = A\cos\sqrt{-\lambda}\,x + B\sin\sqrt{-\lambda}\,x.$$

From the condition $X(0) = 0$, we obtain $A = 0$. The condition $X(l) = 0$ gives

$$B\sin\sqrt{-\lambda}\,l = 0.$$

If $B = 0$, the solution is trivial. For nontrivial solutions, $B \neq 0$, hence,

$$\sin\sqrt{-\lambda}\,l = 0.$$

This equation is satisfied when

$$\sqrt{-\lambda}\,l = n\pi \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

or

$$-\lambda_n = (n\pi/l)^2. \tag{7.3.14}$$

For this infinite set of discrete values of λ, the problem has a nontrivial solution. These values of λ_n are called the *eigenvalues* of the problem, and the functions

$$\sin(n\pi/l)\,x, \qquad n = 1, 2, 3, \ldots$$

are the corresponding *eigenfunctions*.

We note that it is not necessary to consider negative values of n since

$$\sin(-n)\,\pi x/l = -\sin n\pi x/l.$$

No new solution is obtained in this way.

The solutions of problems (7.3.12) are, therefore,

$$X_n(x) = B_n \sin(n\pi x/l). \tag{7.3.15}$$

For $\lambda = \lambda_n$, the general solution of equation (7.3.9) may be written in the form

$$T_n(t) = C_n \cos\left(\frac{n\pi c}{l}\right)t + D_n \sin\left(\frac{n\pi c}{l}\right)t, \tag{7.3.16}$$

where C_n and D_n are arbitrary constants.

Thus, the functions

$$u_n(x,t) = X_n(x)\,T_n(t) = \left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right) \tag{7.3.17}$$

satisfy equation (7.3.1) and the boundary conditions (7.3.4) and (7.3.5), where $a_n = B_n C_n$ and $b_n = B_n D_n$.

Since equation (7.3.1) is linear and homogeneous, by the superposition principle, the infinite series

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right) \tag{7.3.18}$$

is also a solution, provided it converges and is twice continuously differentiable with respect to x and t. Since each term of the series satisfies the boundary conditions (7.3.4) and (7.3.5), the series satisfies these conditions. There remain two more initial conditions to be satisfied. From these conditions, we shall determine the constants a_n and b_n.

First we differentiate the series (7.3.18) with respect to t. We have

$$u_t = \sum_{n=1}^{\infty}\frac{n\pi c}{l}\left(-a_n \sin\frac{n\pi c}{l}t + b_n \cos\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right). \tag{7.3.19}$$

Then applying the initial conditions (7.3.2) and (7.3.3), we obtain

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right), \tag{7.3.20}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right). \tag{7.3.21}$$

These equations will be satisfied if $f(x)$ and $g(x)$ can be represented by Fourier sine series. The coefficients are given by

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \qquad b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx, \tag{7.3.22ab}$$

Please see Fourier series formula

The solution of the vibrating string problem is therefore given by the series (7.3.18) where the coefficients a_n and b_n are determined by the formulae (7.3.22ab).

We examine the physical significance of the solution (7.3.17) in the context of the free vibration of a string of length l. The eigenfunctions

$$u_n(x,t) = (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin\left(\frac{n\pi x}{l}\right), \quad \omega_n = \frac{n\pi c}{l}, \quad (7.3.23)$$

are called the nth *normal modes* of vibration or the nth harmonic, and ω_n represent the discrete spectrum of *circular* (or *radian*) *frequencies* or $\nu_n = \frac{\omega_n}{2\pi} = \frac{nc}{2l}$, which are called the *angular frequencies.* The first harmonic ($n = 1$) is called the *fundamental harmonic* and all other harmonics ($n > 1$) are called *overtones.* The frequency of the fundamental mode is given by

$$\omega_1 = \frac{\pi c}{l}, \qquad \nu_1 = \frac{1}{2l}\sqrt{\frac{T^*}{\rho}}. \quad (7.3.24)$$

Result (7.3.24) is considered the fundamental law (or *Mersenne law*) of a stringed musical instrument. The angular frequency of the fundamental mode of transverse vibration of a string varies as the square root of the tension, inversely as length, and inversely as the square root of the density. The period of the fundamental mode is $T_1 = \frac{2c}{\omega_1} = \frac{2l}{c}$, which is called the fundamental period. Finally, the solution (7.3.18) describes the motion of a plucked string as a superposition of all normal modes of vibration with frequencies which are all integral multiples ($\omega_n = n\omega_1$ or $\nu_n = n\nu_1$) of the fundamental frequency. This is the main reason that stringed instruments produce sweeter musical sounds (or tones) than drum instruments.

In order to describe waves produced in the plucked string with zero initial velocity ($u_t(x,0) = 0$), we write the solution (7.3.23) in the form

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right), \quad n = 1, 2, 3, \ldots. \quad (7.3.25)$$

These solutions are called *standing waves* with amplitude $a_n \sin\left(\frac{n\pi x}{l}\right)$, which vanishes at

$$x = 0,\ \frac{l}{n},\ \frac{2l}{n}, \ldots, l.$$

These are called the *nodes* of the nth harmonic. The string displays n loops separated by the nodes as shown in Figure 7.3.1.

It follows from elementary trigonometry that (7.3.25) takes the form

$$u_n(x,t) = \frac{1}{2} a_n \left[\sin \frac{n\pi}{l}(x - ct) + \sin \frac{n\pi}{l}(x + ct)\right]. \quad (7.3.26)$$

This shows that a standing wave is expressed as a sum of two progressive waves of equal amplitude traveling in opposite directions. This result is in agreement with the d'Alembert solution.

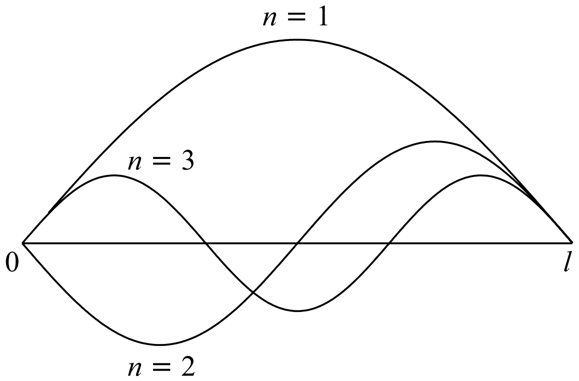

Figure 7.3.1 Several modes of vibration in a string.

Finally, we can rewrite the solution (7.3.23) of the nth normal modes in the form

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l} - \varepsilon_n\right), \tag{7.3.27}$$

where $c_n = \left(a_n^2 + b_n^2\right)^{\frac{1}{2}}$ and $\tan\,\varepsilon_n = \left(\frac{b_n}{a_n}\right)$.

This solution represents transverse vibrations of the string at any point x and at any time t with amplitude $c_n \sin\left(\frac{n\pi x}{l}\right)$ and circular frequency $\omega_n = \frac{n\pi c}{l}$. This form of the solution enables us to calculate the kinetic and potential energies of the transverse vibrations. The total kinetic energy (K.E.) is obtained by integrating with respect to x from 0 to l, that is,

$$K_n = K.E. = \int_0^l \frac{1}{2}\rho\left(\frac{\partial u_n}{\partial t}\right)^2 dx, \tag{7.3.28}$$

where ρ is the line density of the string. Similarly, the total potential energy (P.E.) is given by

$$V_n = P.E. = \frac{1}{2}T^* \int_0^l \left(\frac{\partial u_n}{\partial x}\right)^2 dx. \tag{7.3.29}$$

Substituting (7.3.27) in (7.3.28) and (7.3.29) gives

$$\begin{aligned} K_n &= \frac{1}{2}\rho\left(\frac{n\pi c}{l}c_n\right)^2 \sin^2\left(\frac{n\pi ct}{l} - \varepsilon_n\right)\int_0^l \sin^2\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{\rho c^2\pi^2}{4l}\left(n\,c_n\right)^2 \sin^2\left(\frac{n\pi ct}{l} - \varepsilon_n\right) = \frac{1}{4}\rho l \omega_n^2 c_n^2 \sin^2\left(\omega_n t - \varepsilon_n\right), \end{aligned} \tag{7.3.30}$$

where $\omega_n = \frac{n\pi c}{l}$.

Similarly,

$$V_n = \frac{1}{2} T^* \left(\frac{n\pi c_n}{l}\right)^2 \cos^2\left(\frac{n\pi ct}{l} - \varepsilon_n\right) \int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx$$
$$= \frac{\pi^2 T^*}{4l} (n\, c_n)^2 \cos^2\left(\frac{n\pi ct}{l} - \varepsilon_n\right) = \frac{1}{4}\rho l \omega_n^2 c_n^2 \cos^2(\omega_n t - \varepsilon_n). \quad (7.3.31)$$

Thus, the total energy of the nth normal mode of vibrations is given by

$$E_n = K_n + V_n = \frac{1}{4}\rho l (\omega_n c_n)^2 = \text{constant}. \quad (7.3.32)$$

For a given string oscillating in a normal mode, the total energy is proportional to the square of the circular frequency and to the square of the amplitude.

Finally, the total energy of the system is given by

$$E = \sum_{n=1}^{\infty} E_n = \frac{1}{4}\rho l \sum_{n=1}^{\infty} \omega_n^2 c_n^2, \quad (7.3.33)$$

which is constant because $E_n =$ constant.

Example 7.3.1. The Plucked String of length l

As a special case of the problem just treated, consider a stretched string fixed at both ends. Suppose the string is raised to a height h at $x = a$ and then released. The string will oscillate freely. The initial conditions, as shown in Figure 7.3.2, may be written

Solve this example

$$u(x, 0) = f(x) = \begin{cases} hx/a, & 0 \le x \le a \\ h(l - x)/(l - a), & a \le x \le l. \end{cases}$$

Since $g(x) = 0$, the coefficients b_n are identically equal to zero. The coefficients a_n, according to equation (7.3.22a), are given by

$$a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$
$$= \frac{2}{l} \int_0^a \frac{hx}{a} \sin\left(\frac{n\pi x}{l}\right) dx + \frac{2}{l} \int_a^l \frac{h(l - x)}{(l - a)} \sin\left(\frac{n\pi x}{l}\right) dx.$$

Integration and simplification yields

$$a_n = \frac{2hl^2}{\pi^2 a (l - a)} \frac{1}{n^2} \sin\left(\frac{n\pi a}{l}\right).$$

Thus, the displacement of the plucked string is

$$u(x, t) = \frac{2hl^2}{\pi^2 a (l - a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c}{l}\right) t.$$

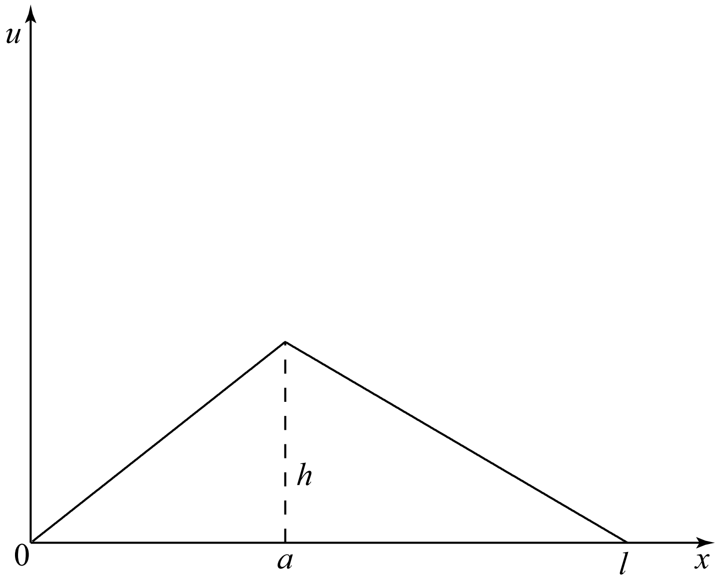

Figure 7.3.2 Plucked String

Solve

Example 7.3.2. The struck string of length l

Here, we consider the string with no initial displacement. Let the string be struck at $x = a$ so that the initial velocity is given by

$$u_t(x,0) = \begin{cases} \frac{v_0}{a}x, & 0 \leq x \leq a \\ v_0(l-x)/(l-a), & a \leq x \leq l \end{cases}.$$

Since $u(x,0) = 0$, we have $a_n = 0$. By applying equation (7.3.22b), we find that

$$\begin{aligned} b_n &= \frac{2}{n\pi c}\int_0^a \frac{v_0}{a}\, x \sin\left(\frac{n\pi x}{l}\right) dx + \frac{2}{n\pi c}\int_a^l v_0 \frac{(l-x)}{(l-a)} \sin\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2v_0 l^3}{\pi^3 ca(l-a)} \frac{1}{n^3} \sin\left(\frac{n\pi a}{l}\right). \end{aligned}$$

Hence, the displacement of the struck string is

$$u(x,t) = \frac{2v_0 l^3}{\pi^3 ca(l-a)} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi a}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c}{l}\right) t.$$

7.4 Existence and Uniqueness of Solution of the Vibrating String Problem

In the preceding section we found that the initial boundary-value problem (7.3.1)–(7.3.5) has a formal solution given by (7.3.18). We shall now show that the expression (7.3.18) is the solution of the problem under certain conditions.

First we see that

$$u_1(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right) \tag{7.4.1}$$

is the formal solution of the problem (7.3.1)–(7.3.5) with $g(x) \equiv 0$, and

$$u_2(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right) \tag{7.4.2}$$

is the formal solution of the above problem with $f(x) \equiv 0$. By linearity of the problem, the solution (7.3.18) may be considered as the sum of the two formal solutions (7.4.1) and (7.4.2).

We first assume that $f(x)$ and $f'(x)$ are continuous on $[0,l]$, and $f(0) = f(l) = 0$. Then by Theorem 6.10.1, the series for the function $f(x)$ given by (7.3.20) converges absolutely and uniformly on the interval $[0,l]$.

Using the trigonometric identity

$$\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi c}{l}t\right) = \frac{1}{2}\sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sin\frac{n\pi}{l}(x+ct), \tag{7.4.3}$$

$u_1(x,t)$ may be written as

$$u_1(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}(x+ct).$$

Define

$$F(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \tag{7.4.4}$$

and assume that $F(x)$ is the odd periodic extension of $f(x)$, that is,

$$\begin{aligned} F(x) &= f(x) \quad 0 \le x \le l \\ F(-x) &= -F(x) \quad \text{for all } x \\ F(x \pm 2l) &= F(x). \end{aligned}$$

We can now rewrite $u_1(x,t)$ in the form

$$u_1(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]. \tag{7.4.5}$$

To show that the boundary conditions are satisfied, we note that

$$\begin{aligned}
u_1(0,t) &= \frac{1}{2}\left[F(-ct) + F(ct)\right] \\
&= \frac{1}{2}\left[-F(ct) + F(ct)\right] = 0 \\
u_1(l,t) &= \frac{1}{2}\left[F(l-ct) + F(l+ct)\right] \\
&= \frac{1}{2}\left[F(-l-ct) + F(l+ct)\right] \\
&= \frac{1}{2}\left[-F(l+ct) + F(l+ct)\right] = 0.
\end{aligned}$$

Since

$$\begin{aligned}
u_1(x,0) &= \frac{1}{2}\left[F(x) + F(x)\right] \\
&= F(x) = f(x), \qquad 0 \le x \le l,
\end{aligned}$$

we see that the initial condition $u_1(x,0) = f(x)$ is satisfied. Thus, equation (7.3.1) and conditions (7.3.2)–(7.3.3) with $g(x) \equiv 0$ are satisfied. Since f' is continuous in $[0,l]$, F' exists and is continuous for all x. Thus, if we differentiate $u_1(x,t)$ with respect to t, we obtain

$$\frac{\partial u_1}{\partial t} = \frac{1}{2}\left[-c\,F'(x-ct) + c\,F'(x+ct)\right],$$

and

$$\frac{\partial u_1}{\partial t}(x,0) = \frac{1}{2}\left[-c\,F'(x) + c\,F'(x)\right] = 0.$$

We therefore see that initial condition (7.3.3) is also satisfied.

In order to show that $u_1(x,t)$ satisfies the differential equation (7.3.1), we impose additional restrictions on f. Let f'' be continuous on $[0,l]$ and $f''(0) = f''(l) = 0$. Then, F'' exists and is continuous everywhere, and therefore,

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial t^2} &= \frac{1}{2}c^2\left[F''(x-ct) + F''(x+ct)\right], \\
\frac{\partial^2 u_1}{\partial x^2} &= \frac{1}{2}\left[F''(x-ct) + F''(x+ct)\right].
\end{aligned}$$

We find therefore that

$$\frac{\partial^2 u_1}{\partial t^2} = c^2\frac{\partial^2 u_1}{\partial x^2}.$$

Next, we shall state the assumptions which must be imposed on g to make $u_2(x,t)$ the solution of problem (7.3.1)–(7.3.5) with $f(x) \equiv 0$. Let g

and g' be continuous on $[0,l]$ and let $g(0) = g(l) = 0$. Then the series for the function $g(x)$ given by (7.3.21) converges absolutely and uniformly in the interval $[0,l]$. Introducing the new coefficients $c_n = (n\pi c/l)\, b_n$, we have

$$u_2(x,t) = \left(\frac{l}{\pi c}\right)\sum_{n=1}^{\infty}\frac{c_n}{n}\sin\left(\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right). \tag{7.4.6}$$

We shall see that term-by-term differentiation with respect to t is permitted, and hence,

$$\frac{\partial u_2}{\partial t} = \sum_{n=1}^{\infty} c_n\cos\left(\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right). \tag{7.4.7}$$

Using the trigonometric identity (7.4.3), we obtain

$$\frac{\partial u_2}{\partial t} = \frac{1}{2}\sum_{n=1}^{\infty} c_n\sin\frac{n\pi}{l}(x-ct) + \frac{1}{2}\sum_{n=1}^{\infty} c_n\sin\frac{n\pi}{l}(x+ct). \tag{7.4.8}$$

These series are absolutely and uniformly convergent because of the assumptions on g, and hence, the series (7.4.6) and (7.4.7) converge absolutely and uniformly on $[0,l]$. Thus, the term-by-term differentiation is justified.

Let

$$G(x) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi x}{l}\right)$$

be the odd periodic extension of the function $g(x)$. Then, equation (7.4.8) can be written in the form

$$\frac{\partial u_2}{\partial t} = \frac{1}{2}\left[G(x-ct) + G(x+ct)\right].$$

Integration yields

$$\begin{aligned} u_2(x,t) &= \frac{1}{2}\int_0^t G(x-ct')\,dt' + \frac{1}{2}\int_0^t G(x+ct')\,dt' \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} G(\tau)\,d\tau. \end{aligned} \tag{7.4.9}$$

It immediately follows that $u_2(x,0) = 0$, and

$$\frac{\partial u_2}{\partial t}(x,0) = G(x) = g(x), \quad 0 \le x \le l.$$

Moreover,

$$\begin{aligned} u_2(0,t) &= \frac{1}{2}\int_0^t G(-ct')\,dt' + \frac{1}{2}\int_0^t G(ct')\,dt' \\ &= -\frac{1}{2}\int_0^t G(ct')\,dt' + \frac{1}{2}\int_0^t G(ct')\,dt' = 0 \end{aligned}$$

and

$$\begin{aligned} u_2(l,t) &= \frac{1}{2}\int_0^t G(l-ct')\,dt' + \frac{1}{2}\int_0^t G(l+ct')\,dt' \\ &= \frac{1}{2}\int_0^t G(-l-ct')\,dt' + \frac{1}{2}\int_0^t G(l+ct')\,dt' \\ &= -\frac{1}{2}\int_0^t G(l+ct')\,dt' + \frac{1}{2}\int_0^t G(l+ct')\,dt' = 0. \end{aligned}$$

Finally, $u_2(x,t)$ must satisfy the differential equation. Since g' is continuous on $[0,l]$, G' exists so that

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{c}{2}\left[-G'(x-ct) + G'(x+ct)\right].$$

Differentiating $u_2(x,t)$ represented by equation (7.4.6) with respect to x, we obtain

$$\begin{aligned} \frac{\partial u_2}{\partial x} &= \frac{1}{c}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi c}{l}t\right)\cos\left(\frac{n\pi x}{l}\right) \\ &= \frac{1}{2c}\sum_{n=1}^{\infty} c_n\left[-\sin\frac{n\pi}{l}(x-ct) + \sin\frac{n\pi}{l}(x+ct)\right] \\ &= \frac{1}{2c}\left[-G(x-ct) + G(x+ct)\right]. \end{aligned}$$

Differentiating again with respect to x, we obtain

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{1}{2c}\left[-G'(x-ct) + G'(x+ct)\right].$$

It is quite evident that

$$\frac{\partial^2 u_2}{\partial t^2} = c^2\frac{\partial^2 u_2}{\partial x^2}.$$

Thus, the solution of the initial boundary-value problem (7.3.1)–(7.3.5) is established.

Theorem 7.4.2. ***(Uniqueness Theorem)*** *There exists at most one solution of the wave equation*

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \quad t > 0,$$

satisfying the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \ge 0,$$

where $u(x,t)$ is a twice continuously differentiable function with respect to both x and t.

Proof. Suppose that there are two solutions u_1 and u_2 and let $v = u_1 - u_2$. It can readily be seen that $v(x,t)$ is the solution of the problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, && 0 < x < l, \quad t > 0, \\ v(0,t) &= 0, && t \geq 0, \\ v(l,t) &= 0, && t \geq 0, \\ v(x,0) &= 0, && 0 \leq x \leq l, \\ v_t(x,0) &= 0, && 0 \leq x \leq l. \end{aligned}$$

We shall prove that the function $v(x,t)$ is identically zero. To do so, consider the energy integral

$$E(t) = \frac{1}{2} \int_0^l \left(c^2 v_x^2 + v_t^2\right) dx \tag{7.4.10}$$

which physically represents the total energy of the vibrating string at time t.

Since the function $v(x,t)$ is twice continuously differentiable, we differentiate $E(t)$ with respect to t. Thus,

$$\frac{dE}{dt} = \int_0^l \left(c^2 v_x v_{xt} + v_t v_{tt}\right) dx. \tag{7.4.11}$$

Integrating the first integral in (7.4.11) by parts, we have

$$\int_0^l c^2 v_x v_{xt} dx = \left[c^2 v_x v_t\right]_0^l - \int_0^l c^2 v_t v_{xx} dx.$$

But from the condition $v(0,t) = 0$ we have $v_t(0,t) = 0$, and similarly, $v_t(l,t) = 0$ for $x = l$. Hence, the expression in the square brackets vanishes, and equation (7.4.11) becomes

$$\frac{dE}{dt} = \int_0^l v_t \left(v_{tt} - c^2 v_{xx}\right) dx. \tag{7.4.12}$$

Since $v_{tt} - c^2 v_{xx} = 0$, equation (7.4.12) reduces to

$$\frac{dE}{dt} = 0$$

which means

$$E(t) = \text{constant} = C.$$

Since $v(x,0) = 0$ we have $v_x(x,0) = 0$. Taking into account the condition $v_t(x,0) = 0$, we evaluate C to obtain

$$E(0) = C = \frac{1}{2}\int_0^l \left[c^2 v_x^2 + v_t^2\right]_{t=0} dx = 0.$$

This implies that $E(t) = 0$ which can happen only when $v_x = 0$ and $v_t = 0$ for $t > 0$. To satisfy both of these conditions, we must have $v(x,t) =$ constant. Employing the condition $v(x,0) = 0$, we then find $v(x,t) = 0$. Therefore, $u_1(x,t) = u_2(x,t)$ and the solution $u(x,t)$ is unique.

7.5 The Heat Conduction Problem

We consider a homogeneous rod of length l. The rod is sufficiently thin so that the heat is distributed equally over the cross section at time t. The surface of the rod is insulated, and therefore, there is no heat loss through the boundary. The temperature distribution of the rod is given by the solution of the initial boundary-value problem

$$\begin{aligned} u_t &= k u_{xx}, \quad 0 < x < l, \quad t > 0, \\ u(0,t) &= 0, \qquad\qquad\qquad t \ge 0, \\ u(l,t) &= 0, \qquad\qquad\qquad t \ge 0, \\ u(x,0) &= f(x), \quad 0 \le x \le l. \end{aligned} \tag{7.5.1}$$

If we assume a solution in the form

$$u(x,t) = X(x)\,T(t) \neq 0.$$

Equation (7.5.1) yields

$$XT' = kX''T.$$

Thus, we have

$$\frac{X''}{X} = \frac{T'}{kT} = -\alpha^2,$$

where α is a positive constant. Hence, X and T must satisfy

$$X'' + \alpha^2 X = 0, \tag{7.5.2}$$

$$T' + \alpha^2 k T = 0. \tag{7.5.3}$$

From the boundary conditions, we have

$$u(0,t) = X(0)\,T(t) = 0, \qquad u(l,t) = X(l)\,T(t) = 0.$$

Thus,

$$X(0) = 0, \qquad X(l) = 0,$$

for an arbitrary function $T(t)$. Hence, we must solve the eigenvalue problem

$$X'' + \alpha^2 X = 0,$$
$$X(0) = 0, \qquad X(l) = 0.$$

The solution of equation (7.5.2) is

$$X(x) = A\cos\alpha x + B\sin\alpha x.$$

Since $X(0) = 0$, $A = 0$. To satisfy the second condition, we have

$$X(l) = B\sin\alpha l = 0.$$

Since $B = 0$ yields a trivial solution, we must have $B \neq 0$ and hence,

$$\sin\alpha l = 0.$$

Thus,

$$\alpha = \frac{n\pi}{l} \quad \text{for} \quad n = 1, 2, 3 \ldots.$$

Substituting these eigenvalues, we have

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right).$$

Next, we consider equation (7.5.3), namely,

$$T' + \alpha^2 kT = 0,$$

the solution of which is

$$T(t) = Ce^{-\alpha^2 kt}.$$

Substituting $\alpha = (n\pi/l)$, we have

$$T_n(t) = C_n e^{-(n\pi/l)^2 kt}.$$

Hence, the nontrivial solution of the heat equation which satisfies the two boundary conditions is

$$u_n(x,t) = X_n(x)\,T_n(t) = a_n\, e^{-(n\pi/l)^2 kt}\, \sin\left(\frac{n\pi x}{l}\right), \qquad n = 1, 2, 3 \ldots,$$

where $a_n = B_n C_n$ is an arbitrary constant.

By the principle of superposition, we obtain a formal series solution as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t),$$
$$= \sum_{n=1}^{\infty} a_n\, e^{-(n\pi/l)^2 kt}\, \sin\left(\frac{n\pi x}{l}\right), \tag{7.5.4}$$

which satisfies the initial condition if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right).$$

This holds true if $f(x)$ can be represented by a Fourier sine series with Fourier coefficients

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \tag{7.5.5}$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(\tau)\sin\left(\frac{n\pi\tau}{l}\right)d\tau\right]e^{-(n\pi/l)^2 kt}\sin\left(\frac{n\pi x}{l}\right) \tag{7.5.6}$$

is the formal series solution of the heat conduction problem.

Solve

Example 7.5.1. (a) Suppose the initial temperature distribution is $f(x) = x(l-x)$. Then, from equation (7.5.5), we have

$$a_n = \frac{8l^2}{n^3\pi^3}, \qquad n = 1, 3, 5, \ldots.$$

Thus, the solution is

$$u(x,t) = \left(\frac{8l^2}{\pi^3}\right)\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}\,e^{-(n\pi/l)^2 kt}\sin\left(\frac{n\pi x}{l}\right).$$

(b) Suppose the temperature at one end of the rod is held constant, that is,

$$u(l,t) = u_0, \qquad t \geq 0.$$

The problem here is

$$\begin{aligned} u_t &= k\,u_{xx}, && 0 < x < l, \quad t > 0,\\ u(0,t) &= 0, && u(l,t) = u_0,\\ u(x,0) &= f(x), && 0 < x < l. \end{aligned} \tag{7.5.7}$$

Let

$$u(x,t) = v(x,t) + \frac{u_0 x}{l}.$$

Substitution of $u(x,t)$ in equations (7.5.7) yields

$$\begin{aligned} v_t &= k\,v_{xx}, && 0 < x < l, \quad t > 0,\\ v(0,t) &= 0, && v(l,t) = 0,\\ v(x,0) &= f(x) - \frac{u_0 x}{l}, && 0 < x < l. \end{aligned}$$

Hence, with the knowledge of solution (7.5.6), we obtain the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l} \int_0^l \left(f(\tau) - \frac{u_0 \tau}{l} \right) \sin\left(\frac{n\pi\tau}{l} \right) d\tau \right] e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l} \right) + \left(\frac{u_0 x}{l} \right). \quad (7.5.8)$$

7.6 Existence and Uniqueness of Solution of the Heat Conduction Problem

In the preceding section, we found that (7.5.4) is the formal solution of the heat conduction problem (7.5.1), where a_n is given by (7.5.5).

We shall prove the existence of this formal solution if $f(x)$ is continuous in $[0, l]$ and $f(0) = f(l) = 0$, and $f'(x)$ is piecewise continuous in $(0, l)$. Since $f(x)$ is bounded, we have

$$|a_n| = \frac{2}{l} \left| \int_0^l f(x) \sin\left(\frac{n\pi x}{l} \right) dx \right| \leq \frac{2}{l} \int_0^l |f(x)|\, dx \leq C,$$

where C is a positive constant. Thus, for any finite $t_0 > 0$,

$$\left| a_n\, e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l} \right) \right| \leq C\, e^{-(n\pi/l)^2 kt_0} \quad \text{when} \quad t \geq t_0.$$

According to the ratio test, the series of terms $\exp\left[-(n\pi/l)^2 kt_0 \right]$ converges. Hence, by the Weierstrass M-test, the series (7.5.4) converges uniformly with respect to x and t whenever $t \geq t_0$ and $0 \leq x \leq l$.

Differentiating equation (7.5.4) termwise with respect to t, we obtain

$$u_t = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l} \right)^2 k\, e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l} \right). \quad (7.6.1)$$

We note that

$$\left| -a_n \left(\frac{n\pi}{l} \right)^2 k\, e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l} \right) \right| \leq C \left(\frac{n\pi}{l} \right)^2 k\, e^{-(n\pi/l)^2 kt_0}$$

when $t \geq t_0$, and the series of terms $C(n\pi/l)^2 k \exp\left[-(n\pi/l)^2 kt_0 \right]$ converges by the ratio test. Hence, equation (7.6.1) is uniformly convergent in the region $0 \leq x \leq l$, $t \geq t_0$. In a similar manner, the series (7.5.4) can be differentiated twice with respect to x, and as a result

$$u_{xx} = -\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{l} \right)^2 e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l} \right). \quad (7.6.2)$$

Evidently, from equations (7.6.1) and (7.6.2),

$$u_t = k\, u_{xx}.$$

Hence, equation (7.5.4) is a solution of the one-dimensional heat equation in the region $0 \le x \le l$, $t \ge 0$.

Next, we show that the boundary conditions are satisfied. Here, we note that the series (7.5.4) representing the function $u\,(x,t)$ converges uniformly in the region $0 \le x \le l$, $t \ge 0$. Since the function represented by a uniformly convergent series of continuous functions is continuous, $u\,(x,t)$ is continuous at $x = 0$ and $x = l$. As a consequence, when $x = 0$ and $x = l$, solution (7.5.4) satisfies

$$u\,(0,t) = 0, \qquad u\,(l,t) = 0,$$

for all $t > 0$.

It remains to show that $u\,(x,t)$ satisfies the initial condition

$$u\,(x,0) = f\,(x)\,, \qquad 0 \le x \le l.$$

Under the assumptions stated earlier, the series for $f\,(x)$ given by

$$f\,(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

is uniformly and absolutely convergent. By Abel's test of convergence the series formed by the product of the terms of a uniformly convergent series

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

and a uniformly bounded and monotone sequence $\exp\left[-\,(n\pi/l)^2\,kt\right]$ converges uniformly with respect to t. Hence,

$$u\,(x,t) = \sum_{n=1}^{\infty} a_n\, e^{-(n\pi/l)^2 kt}\, \sin\left(\frac{n\pi x}{l}\right)$$

converges uniformly for $0 \le x \le l$, $t \ge 0$, and by the same reasoning as before, $u\,(x,t)$ is continuous for $0 \le x \le l$, $t \ge 0$. Thus, the initial condition

$$u\,(x,0) = f\,(x)\,, \qquad 0 \le x \le l$$

is satisfied. The existence of solution is therefore established.

In the above discussion the condition imposed on $f\,(x)$ is stronger than necessary. The solution can be obtained with a less stringent condition on $f\,(x)$ (see Weinberger (1965)).

Theorem 7.6.1. ***(Uniqueness Theorem)*** *Let $u(x,t)$ be a continuously differentiable function. If $u(x,t)$ satisfies the differential equation*

$$u_t = k\,u_{xx}, \quad 0 < x < l, \quad t > 0,$$

the initial conditions

$$u(x,0) = f(x), \quad 0 \le x \le l,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t \ge 0,$$

then, the solution is unique.

Proof. Suppose that there are two distinct solutions $u_1(x,t)$ and $u_2(x,t)$. Let

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

Then,

$$\begin{aligned} v_t &= k\,v_{xx}, & 0 < x < l, && t > 0, \\ v(0,t) &= 0, & v(l,t) = 0, && t \ge 0, \\ v(x,0) &= 0, & 0 \le x \le l, && \end{aligned} \tag{7.6.3}$$

Consider the function defined by the integral

$$J(t) = \frac{1}{2k}\int_0^l v^2 dx.$$

Differentiating with respect to t, we have

$$J'(t) = \frac{1}{k}\int_0^l v v_t dx = \int_0^l v v_{xx} dx,$$

by virtue of equation (7.6.3). Integrating by parts, we have

$$\int_0^l v v_{xx} dx = \left[v v_x\right]_0^l - \int_0^l v_x^2 dx.$$

Since $v(0,t) = v(l,t) = 0$,

$$J'(t) = -\int_0^l v_x^2\, dx \le 0.$$

From the condition $v(x,0) = 0$, we have $J(0) = 0$. This condition and $J'(t) \le 0$ implies that $J(t)$ is a nonincreasing function of t. Thus,

$$J(t) \leq 0.$$

But by definition of $J(t)$,

$$J(t) \geq 0.$$

Hence,

$$J(t) = 0, \quad \text{for } t \geq 0.$$

Since $v(x,t)$ is continuous, $J(t) = 0$ implies

$$v(x,t) = 0$$

in $0 \leq x \leq l$, $t \geq 0$. Therefore, $u_1 = u_2$ and the solution is unique.

7.7 The Laplace and Beam Equations

Example 7.7.1. Consider the steady state temperature distribution in a thin rectangular slab. Two sides are insulated, one side is maintained at zero temperature, and the temperature of the remaining side is prescribed to be $f(x)$. Thus, we are required to solve

$$\begin{aligned} \nabla^2 u &= 0, \qquad 0 < x < a, \quad 0 < y < b, \\ u(x,0) &= f(x), \ 0 \leq x \leq a, \\ u(x,b) &= 0, \qquad 0 \leq x \leq a, \\ u_x(0,y) &= 0, \ u_x(a,y) = 0. \end{aligned}$$

Let $u(x,y) = X(x) Y(y)$. Substitution of this into the Laplace equation yields

$$X'' - \lambda X = 0, \qquad Y'' + \lambda X = 0.$$

Since the boundary conditions are homogeneous on $x = 0$ and $x = a$, we have $\lambda = -\alpha^2$ with $\alpha \geq 0$ for nontrivial solutions of the eigenvalue problem

$$\begin{aligned} X'' + \alpha^2 X &= 0, \\ X'(0) = X'(a) &= 0. \end{aligned}$$

The solution is

$$X(x) = A \cos \alpha x + B \sin \alpha x.$$

Application of the boundary conditions then yields $B = 0$ and $\alpha = (n\pi/a)$ with $n = 0, 1, 2, \ldots$. Hence,

$$X_n(x) = A \cos\left(\frac{n\pi x}{a}\right).$$

The solution of the Y equation is clearly

$$Y(y) = C\cosh\alpha y + D\sinh\alpha y$$

which can be written in the form

$$Y(y) = E\,\sinh\alpha(y+F),$$

where $E = \left(D^2 - C^2\right)^{\frac{1}{2}}$ and $F = \left[\tanh^{-1}(C/D)\right]/\alpha$.

Applying the homogeneous boundary condition $Y(b) = 0$, we obtain

$$Y(b) = E\sinh\alpha(b+F) = 0$$

which implies

$$F = -b, \qquad E \neq 0$$

for nontrivial solutions. Hence, we have

$$u(x,y) = \frac{(b-y)}{b}\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)\sinh\left\{\frac{n\pi}{a}(y-b)\right\}.$$

Now we apply the remaining nonhomogeneous condition to obtain

$$u(x,0) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)\sinh\left(-\frac{n\pi b}{a}\right).$$

Since this is a Fourier cosine series, the coefficients are given by

$$a_0 = \frac{2}{a}\int_0^a f(x)\,dx,$$

$$a_n = \frac{-2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx, \qquad n = 1, 2, \ldots.$$

Thus, the solution is

$$u(x,y) = \left(\frac{b-y}{b}\right)\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^* \frac{\sinh\frac{n\pi}{a}(b-y)}{\sinh\frac{n\pi b}{a}}\cos\left(\frac{n\pi x}{a}\right),$$

where

$$a_n^* = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx.$$

If, for example $f(x) = x$ in $0 < x < \pi$, $0 < y < \pi$, then we find (note that $a = \pi$)

$$a_0 = \pi, \qquad a_n^* = \frac{2}{\pi n^2}\left[(-1)^n - 1\right], \qquad n = 1, 2, \ldots$$

and hence, the solution has the final form

$$u(x,y) = \frac{1}{2}(\pi - y) + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\frac{\sinh n(\pi - y)}{\sinh n\pi}\cos nx.$$

Example 7.7.2. As another example, we consider the transverse vibration of a beam. The equation of motion is governed by

$$u_{tt} + a^2 u_{xxxx} = 0, \qquad 0 < x < l, \quad t > 0,$$

where $u(x,t)$ is the displacement and a is the physical constant. Note that the equation is of the fourth order in x. Let the initial and boundary conditions be

$$\begin{aligned} u(x,0) &= f(x), && 0 \le x \le l, \\ u_t(x,0) &= g(x), && 0 \le x \le l, \\ u(0,t) &= u(l,t) = 0, && t > 0, \\ u_{xx}(0,t) &= u_{xx}(l,t) = 0, && t > 0. \end{aligned} \tag{7.7.1}$$

The boundary conditions represent the beam being simple supported, that is, the displacements and the bending moments at the ends are zero.

Assume a nontrivial solution in the form

$$u(x,t) = X(x)\,T(t),$$

which transforms the equation of motion into the forms

$$T'' + a^2\alpha^4 T = 0, \qquad X^{(iv)} - \alpha^4 X = 0, \qquad \alpha > 0.$$

The equation for $X(x)$ has the general solution

$$X(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\alpha x + D\sin\alpha x.$$

The boundary conditions require that

$$X(0) = X(l) = 0, \qquad X''(0) = X''(l) = 0.$$

Differentiating X twice with respect to x, we obtain

$$X''(x) = A\alpha^2\cosh\alpha x + B\alpha^2\sinh\alpha x - C\alpha^2\cos\alpha x - D\alpha^2\sin\alpha x.$$

Now applying the conditions $X(0) = X''(0) = 0$, we obtain

$$A + C = 0, \qquad \alpha^2(A - C) = 0,$$

and hence,

$$A = C = 0.$$

The conditions $X(l) = X''(l) = 0$ yield

$$\begin{aligned} B\sinh\alpha l + D\sin\alpha l &= 0, \\ B\sinh\alpha l - D\sin\alpha l &= 0. \end{aligned}$$

These equations are satisfied if

$$B \sinh \alpha l = 0, \qquad D \sin \alpha l = 0.$$

Since $\sinh \alpha l \neq 0$, B must vanish. For nontrivial solutions, $D \neq 0$,

$$\sin \alpha l = 0,$$

and hence,

$$\alpha = \left(\frac{n\pi}{l}\right), \qquad n = 1, 2, 3, \ldots.$$

We then obtain

$$X_n(x) = D_n \sin\left(\frac{n\pi x}{l}\right).$$

The general solution for $T(t)$ is

$$T(t) = E \cos\left(a\alpha^2 t\right) + F \sin\left(a\alpha^2 t\right).$$

Inserting the values of α^2, we obtain

$$T_n(t) = E_n \cos\left\{a\left(\frac{n\pi}{l}\right)^2 t\right\} + F_n \sin\left\{a\left(\frac{n\pi}{l}\right)^2 t\right\}.$$

Thus, the general solution for the transverse vibrations of a beam is

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\left\{a\left(\frac{n\pi}{l}\right)^2 t\right\} + b_n \sin\left\{a\left(\frac{n\pi}{l}\right)^2 t\right\}\right] \sin\left(\frac{n\pi x}{l}\right). \tag{7.7.2}$$

To satisfy the initial condition $u(x,0) = f(x)$, we must have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

from which we find

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{7.7.3}$$

Now the application of the second initial condition gives

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n a \left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right)$$

and hence,

$$b_n = \frac{2}{al}\left(\frac{l}{n\pi}\right)^2 \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{7.7.4}$$

Thus, the solution of the initial boundary-value problem is given by equations (7.7.2)–(7.7.4).

7.8 Nonhomogeneous Problems

The partial differential equations considered so far in this chapter are homogeneous. In practice, there is a very important class of problems involving nonhomogeneous equations. First, we shall illustrate a problem involving a time-independent nonhomogeneous equations.

Example 7.8.1. Consider the initial boundary-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + F(x), \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), \qquad 0 \le x \le l, \\ u_t(x,0) &= g(x), \qquad 0 \le x \le l, \\ u(0,t) &= A, \quad u(l,t) = B, \quad t > 0. \end{aligned} \tag{7.8.1}$$

We assume a solution in the form

$$u(x,t) = v(x,t) + U(x).$$

Substitution of $u(x,t)$ in equation (7.8.1) yields

$$v_{tt} = c^2 (v_{xx} + U_{xx}) + F(x),$$

and if $U(x)$ satisfies the equation

$$c^2 U_{xx} + F(x) = 0,$$

then $v(x,t)$ satisfies the wave equation

$$v_{tt} = c^2 v_{xx}.$$

In a similar manner, if $u(x,t)$ is inserted in the initial and boundary conditions, we obtain

$$\begin{aligned} u(x,0) &= v(x,0) + U(x) = f(x), \\ u_t(x,0) &= v_t(x,0) \qquad\quad = g(x), \\ u(0,t) &= v(0,t) + U(0) = A, \\ u(l,t) &= v(l,t) + U(l) \;\; = B. \end{aligned}$$

Thus, if $U(x)$ is the solution of the problem

$$\begin{aligned} c^2 U_{xx} + F &= 0, \\ U(0) = A, &\quad U(l) = B, \end{aligned}$$

then $v(x,t)$ must satisfy

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \\ v(x,0) &= f(x) - U(x), \\ v_t(x,0) &= g(x), \\ v(0,t) &= 0, \quad v(l,t) = 0. \end{aligned} \tag{7.8.2}$$

Now $v(x,t)$ can be solved easily since $U(x)$ is known. It can be seen that

$$U(x) = A + (B-A)\frac{x}{l} + \frac{x}{l}\int_0^l \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi\right] d\eta - \int_0^x \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi\right] d\eta.$$

As a specific example, consider the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + h, \quad h \text{ is a constant} \\ u(x,0) &= 0, \qquad u_t(x,0) = 0, \\ u(0,t) &= 0, \qquad u(l,t) = 0. \end{aligned} \tag{7.8.3}$$

Then, the solution of the system

$$\begin{aligned} c^2 U_{xx} + h &= 0, \\ U(0) &= 0, \qquad U(l) = 0, \end{aligned}$$

is

$$U(x) = \frac{h}{2c^2}\left(lx - x^2\right).$$

The function $v(x,t)$ must satisfy

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \\ v(x,0) &= -\frac{h}{2c^2}\left(lx - x^2\right), \quad v_t(x,0) = 0, \\ v(0,t) &= 0, \qquad v(l,t) = 0. \end{aligned}$$

The solution is given (see Section 7.3 with $g(x) = 0$) by

$$v(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi c}{l}t\right) \sin\left(\frac{n\pi x}{l}\right),$$

and the coefficient is

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l \left[-\frac{h}{2c^2}\left(lx - x^2\right)\right] \sin\left(\frac{n\pi x}{l}\right) dx \\ a_n &= -\frac{4l^2 h}{n^3\pi^3 c^2} \qquad \text{for } n \text{ odd} \\ a_n &= 0 \qquad \text{for } n \text{ even.} \end{aligned}$$

The solution of the given initial boundary-value problem is, therefore, given by

$$\begin{aligned} u(x,t) &= v(x,t) + U(x) \\ &= \frac{hx}{2c^2}(l-x) + \sum_{n=1}^{\infty}\left(-\frac{4l^2h}{c^2\pi^3}\right)\frac{\cos(2n-1)(\pi ct/l)}{(2n-1)^3} \\ &\qquad \times \sin(2n-1)(\pi x/l). \end{aligned} \tag{7.8.4}$$

Let us now consider the problem of a finite string with an external force acting on it. If the ends are fixed, we have

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), & 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), & 0 \le x \le l, \\ u_t(x,0) &= g(x), & 0 \le x \le l, \\ u(0,t) &= 0, & u(l,t) = 0, \quad t \ge 0. \end{aligned} \tag{7.8.5}$$

We assume a solution involving the eigenfunctions, $\sin(n\pi x/l)$, of the associated eigenvalue problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{l}\right), \tag{7.8.6}$$

where the functions $u_n(t)$ are to be determined. It is evident that the boundary conditions are satisfied. Let us also assume that

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t)\sin\left(\frac{n\pi x}{l}\right). \tag{7.8.7}$$

Thus,

$$h_n(t) = \frac{2}{l}\int_0^l h(x,t)\sin\left(\frac{n\pi x}{l}\right)dx. \tag{7.8.8}$$

We assume that the series (7.8.6) is convergent. We then find u_{tt} and u_{xx} from (7.8.6) and substitution of these values into (7.8.5) yields

$$\sum_{n=1}^{\infty}\left[u_n''(t) + \lambda_n^2 u_n(t)\right]\sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} h_n(t)\sin\left(\frac{n\pi x}{l}\right),$$

where $\lambda_n = (n\pi c/l)$. Multiplying both sides of this equation by $\sin(m\pi x/l)$, where $m = 1, 2, 3, \ldots$, and integrating from $x = 0$ to $x = l$, we obtain

$$u_n''(t) + \lambda_n^2 u_n(t) = h_n(t)$$

the solution of which is given by

$$u_n(t) = a_n\cos\lambda_n t + b_n\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t h_n(\tau)\sin\left[\lambda_n(t-\tau)\right]d\tau. \tag{7.8.9}$$

Hence, the formal solution (7.8.6) takes the final form

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos \lambda_n t + b_n \sin \lambda_n t + \frac{1}{\lambda_n} \int_0^t h_n(\tau) \sin [\lambda_n (t-\tau)]\, d\tau \right\} \cdot \sin\left(\frac{n\pi x}{l}\right). \quad (7.8.10)$$

Applying the initial conditions, we have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right).$$

Thus,

$$a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \quad (7.8.11)$$

Similarly,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \lambda_n \sin\left(\frac{n\pi x}{l}\right).$$

Thus,

$$b_n = \left(\frac{2}{l\lambda_n}\right) \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx. \quad (7.8.12)$$

Hence, the formal solution of the initial boundary-value problem (7.8.5) is given by (7.8.10) with a_n given by (7.8.11) and b_n given by (7.8.12).

Example 7.8.2. Determine the solution of the initial boundary-value problem

$$\begin{aligned} u_{tt} - u_{xx} &= h, && 0 < x < 1, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= x(1-x), && 0 \le x \le 1, \\ u_t(x,0) &= 0, && 0 \le x \le 1, \\ u(0,t) &= 0, && u(1,t) = 0, \quad t \ge 0. \end{aligned} \quad (7.8.13)$$

In this case, $c = 1$, $\lambda_n = n\pi$, $b_n = 0$ and a_n is given by

$$a_n = 2 \int_0^1 x(1-x) \sin n\pi x\, dx = \frac{4}{(n\pi)^3} [1 - (-1)^n].$$

We also have

$$h_n = 2 \int_0^1 h \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2h}{n\pi} [1 - (-1)^n].$$

Hence, the integral term in (7.8.9) represents $\phi_n(t)$ given by

$$\phi_n(t) = \frac{1}{\lambda_n}\int_0^t h_n(\tau)\sin[\lambda_n(t-\tau)]\,d\tau = \frac{2h}{n\pi\lambda_n^2}[1-(-1)^n](1-\cos\lambda_n t).$$

The solution (7.8.10) is thus given by

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\frac{4}{n^3\pi^3}[1-(-1)^n]\cos n\pi t + \frac{2h}{n^3\pi^3}[1-(-1)^n](1-\cos n\pi t)\right\}\cdot\sin n\pi x. \quad (7.8.14)$$

We have treated the initial boundary-value problem with the fixed end conditions. Problems with other boundary conditions can also be solved in a similar manner.

We will now consider the initial boundary-value problem with time-dependent boundary conditions, namely,

$$\begin{aligned} u_{tt} - u_{xx} &= h(x,t), && 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u_t(x,0) &= g(x), && 0 \le x \le l, \\ u(0,t) &= p(t), && u(l,t) = q(t), \quad t \ge 0. \end{aligned} \quad (7.8.15)$$

We assume a solution in the form

$$u(x,t) = v(x,t) + U(x,t). \quad (7.8.16)$$

Substituting this into equation (7.8.15), we obtain

$$v_{tt} - c^2 v_{xx} = h - U_{tt} + c^2 U_{xx}.$$

For the initial and boundary conditions, we have

$$\begin{aligned} v(x,0) &= f(x) - U(x,0), \\ v_t(x,0) &= g(x) - U_t(x,0), \\ v(0,t) &= p(t) - U(0,t), \\ v(l,t) &= q(t) - U(l,t). \end{aligned}$$

In order to make the boundary conditions homogeneous, we set

$$U(0,t) = p(t), \qquad U(l,t) = q(t).$$

Thus, $U(x,t)$ must take the form

$$U(x,t) = p(t) + \frac{x}{l}[q(t) - p(t)]. \quad (7.8.17)$$

The problem now is to find the function $v(x,t)$ which satisfies

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= h - U_{tt} = H(x,t), \\ v(x,0) &= f(x) - U(x,0) = F(x), \\ v_t(x,0) &= g(x) - U_t(x,0) = G(x), \\ v(0,t) &= 0, \qquad v(l,t) = 0. \end{aligned} \tag{7.8.18}$$

This is the same type of problem as the one with homogeneous boundary condition that has previously been treated.

Example 7.8.3. Find the solution of the problem

$$\begin{aligned} u_{tt} - u_{xx} &= h, && 0 < x < 1, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= x(1-x), && 0 \le x \le 1, \\ u_t(x,0) &= 0, && 0 \le x \le 1, \\ u(0,t) &= t, && u(1,t) = \sin t, \quad t \ge 0. \end{aligned} \tag{7.8.19}$$

In this case, we use (7.8.16) and (7.8.17) with $c = 1$ and $\lambda_n = n\pi$ so that

$$u(x,t) = v(x,t) + U(x,t), \qquad U(x,t) = t + x(\sin t - t). \tag{7.8.20}$$

Then, v must satisfy

$$\begin{aligned} v_{tt} - v_{xx} &= h + x \sin t, \\ v(x,0) &= x(1-x), \\ v_t(x,0) &= -1, \\ v(0,t) &= 0, \quad v(1,t) = 0. \end{aligned} \tag{7.8.21}$$

It follows from (7.8.8) that

$$\begin{aligned} h_n(t) &= 2 \int_0^1 (h + x \sin t) \sin n\pi x \, dx \\ &= \frac{2h}{n\pi}\left[1 - (-1)^n\right] + \frac{2(-1)^{n+1}}{n\pi} \sin t = a + b \sin t \,(\text{say}). \end{aligned} \tag{7.8.22}$$

We also find

$$a_n = 2 \int_0^1 x(1-x) \sin n\pi x \, dx = \frac{4}{(n\pi)^3}\left[1 - (-1)^n\right],$$

and

$$b_n = \frac{2}{n\pi} \int_0^1 \sin n\pi x \, dx = \frac{2}{(n\pi)^2}\left[1 - (-1)^n\right].$$

Then, we determine the integral term in (7.8.9) so that

$$\begin{aligned}\phi_n(t) &= \frac{1}{n\pi}\int_0^t (a+b\sin\tau)\sin[n\pi(t-\tau)]\,d\tau \\ &= \frac{1}{n\pi}\left\{\frac{a}{n\pi}(1-\cos n\pi t)+\frac{b}{4}[(\sin 2t-2t)\cos n\pi t\right. \\ &\qquad \left.-(\cos 2t-1)\sin n\pi t]\right\}. \quad (7.8.23)\end{aligned}$$

Hence, the solution of the problem (7.8.21) is

$$v(x,t)=\sum_{n=1}^{\infty}[a_n\cos n\pi t+b_n\sin n\pi t+\phi_n(t)]\sin n\pi x. \quad (7.8.24)$$

Thus, the solution of problem (7.8.19) is given by

$$u(x,t)=v(x,t)+U(x,t),$$

where $v(x,t)$ is given by (7.8.24) and $U(x,t)$ is given by (7.8.20)

Example 7.8.4. Use the method of separation of variables to derive the Hermite equation from the *Fokker–Planck equation* of nonequilibrium statistical mechanics

$$u_t-u_{xx}=(x\,u)_x\,. \quad (7.8.25)$$

We seek a nontrivial separable solution $u(x,t)=X(x)T(t)$ so that equation (7.8.25) reduces to a pair of ordinary differential equations

$$X''+xX'+(1+n)X=0 \quad \text{and} \quad T'+nT=0, \quad (7.8.26ab)$$

where $(-n)$ is a separation constant.

We next use

$$X(x)=\exp\left(-\frac{1}{2}x^2\right)f(x) \quad (7.8.27)$$

and rescale the independent variable to obtain the Hermite equation for f in the form

$$\frac{d^2f}{d\xi^2}-2\xi\frac{df}{d\xi}+2nf=0.$$

The solution of (7.8.26b) gives

$$T(t)=c_n\exp(-nt), \quad (7.8.28)$$

where the coefficients c_n are constants.

Thus, the solution of the Fokker–Planck equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-nt - \frac{1}{2}x^2\right) H_n\left(\frac{x}{\sqrt{2}}\right), \tag{7.8.29}$$

where H_n is the Hermite function and a_n are arbitrary constants to be determined from the given initial condition

$$u(x,0) = f(x). \tag{7.8.30}$$

We make the change of variables

$$\xi = x\, e^t \quad \text{and} \quad u = e^t v, \tag{7.8.31}$$

in equation (7.8.25). Consequently, equation (7.8.25) becomes

$$\frac{\partial v}{\partial t} = e^{2t}\, \frac{\partial^2 v}{\partial \xi^2}. \tag{7.8.32}$$

Making another change of variable t to $\tau(t)$, we transform (7.8.32) into the linear diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2}. \tag{7.8.33}$$

Finally, we note that the asymptotic behavior of the solution $u(x,t)$ as $t \to \infty$ is of special interest. The reader is referred to Reif (1965) for such behavior.

7.9 Exercises

1. Solve the following initial boundary-value problems:

(a)
$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = x(1-x), \qquad u_t(x,0) = 0, \qquad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = 3\sin x, \qquad u_t(x,0) = 0, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

2. Determine the solutions of the following initial boundary-value problems:

(a)
$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = 8\sin^2 x, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = x\sin \pi x, \qquad 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

3. Find the solution of each of the following problems:

(a)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < 1, \qquad t > 0,$$
$$u(x,0) = x(1-x), \qquad u_t(x,0) = x - \tan \tfrac{\pi x}{4}, \qquad 0 \le x \le 1,$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = \sin x, \qquad u_t(x,0) = x^2 - \pi x, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0.$$

4. Solve the following problems:

(a)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = x + \sin x, \qquad u_t(x,0) = 0, \qquad 0 \le x \le \pi,$$
$$u(0,t) = u_x(\pi,t) = 0, \qquad t > 0.$$

(b)
$$u_{tt} = c^2 u_{xx} = 0, \qquad 0 < x < \pi, \qquad t > 0,$$
$$u(x,0) = \cos x, \qquad u_t(x,0) = 0, \qquad 0 \le x \le \pi,$$
$$u_x(0,t) = 0, \qquad u_x(\pi,t) = 0, \qquad t > 0.$$

5. By the method of separation of variables, solve the telegraph equation:

$$\begin{aligned} u_{tt} + a u_t + b u &= c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0, \\ u(x,0) &= f(x), \qquad u_t(x,0) = 0, \\ u(0,t) &= u(l,t) = 0, \qquad\qquad t > 0. \end{aligned}$$

6. Obtain the solution of the damped wave motion problem:

$$\begin{aligned} u_{tt} + a u_t &= c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0, \\ u(x,0) &= 0, \qquad u_t(x,0) = g(x), \\ u(0,t) &= u(l,t) = 0. \end{aligned}$$

7. The torsional oscillation of a shaft of circular cross section is governed by the partial differential equation

$$\theta_{tt} = a^2 \theta_{xx},$$

where $\theta(x,t)$ is the angular displacement of the cross section and a is a physical constant. The ends of the shaft are fixed elastically, that is,

$$\theta_x(0,t) - h\,\theta(0,t) = 0, \qquad \theta_x(l,t) + h\,\theta(l,t) = 0.$$

Determine the angular displacement if the initial angular displacement is $f(x)$.

8. Solve the initial boundary-value problem of the longitudinal vibration of a truncated cone of length l and base of radius a. The equation of motion is given by

$$\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right], \qquad 0 < x < l, \quad t > 0,$$

where $c^2 = (E/\rho)$, E is the elastic modulus, ρ is the density of the material and $h = la/(a-l)$. The two ends are rigidly fixed. If the initial displacement is $f(x)$, that is, $u(x,0) = f(x)$, find $u(x,t)$.

9. Establish the validity of the formal solution of the initial boundary-value problems:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \pi, \qquad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \pi, \\ u_x(0,t) &= 0, \qquad u_x(\pi,t) = 0, \qquad t > 0. \end{aligned}$$

10. Prove the uniqueness of the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \pi, \qquad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \pi, \\ u_x(0,t) &= 0, \qquad u_x(\pi,t) = 0, \qquad t > 0. \end{aligned}$$

11. Determine the solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + A \sinh x, \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 \le x \le l, \\ u(0,t) &= h, \quad u(l,t) = k, \quad t > 0, \end{aligned}$$

where h, k, and A are constants.

12. Solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + Ax, \quad 0 < x < 1, \quad t > 0, \quad A = \text{constant}, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0. \end{aligned}$$

13. Solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + x^2, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= x, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u(1,t) = 1, \quad t \ge 0. \end{aligned}$$

14. Find the solution of the following problems:

(a)
$$\begin{aligned} u_t &= k u_{xx} + h, \quad 0 < x < 1, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= u_0(1 - \cos \pi x), \quad 0 \le x \le 1, \quad u_0 = \text{constant}, \\ u(0,t) &= 0, \quad u(l,t) = 2u_0, \quad t \ge 0. \end{aligned}$$

(b)
$$\begin{aligned} u_t &= k u_{xx} - hu, \quad 0 < x < l, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= f(x), \quad 0 \le x \le l, \\ u_x(0,t) &= u_x(l,t) = 0, \quad t > 0. \end{aligned}$$

15. Obtain the solution of each of the following initial boundary-value problems:

(a)
$$\begin{aligned} u_t &= 4\, u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= x^2(1-x), \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u(l,t) = 0, \quad t \ge 0. \end{aligned}$$

(b)
$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < \pi, \quad t > 0, \\ u(x,0) &= \sin^2 x, \quad 0 \le x \le \pi, \\ u(0,t) &= 0, \quad u(\pi,t) = 0, \quad t \ge 0. \end{aligned}$$

(c)
$$\begin{aligned} u_t &= u_{xx}, && 0 < x < 2, && t > 0, \\ u(x,0) &= x, && 0 \le x \le 2, && \\ u(0,t) &= 0, && u_x(2,t) = 1, && t \ge 0. \end{aligned}$$

(d)
$$\begin{aligned} u_t &= k\, u_{xx}, && 0 < x < l, && t > 0, \\ u(x,0) &= \sin(\pi x/2l), && 0 \le x \le l, && \\ u(0,t) &= 0, && u(l,t) = 1, && t \ge 0. \end{aligned}$$

16. Find the temperature distribution in a rod of length l. The faces are insulated, and the initial temperature distribution is given by $x(l-x)$.

17. Find the temperature distribution in a rod of length π, one end of which is kept at zero temperature and the other end of which loses heat at a rate proportional to the temperature at that end $x = \pi$. The initial temperature distribution is given by $f(x) = x$.

18. The voltage distribution in an electric transmission line is given by

$$v_t = k\, v_{xx}, \quad 0 < x < l, \quad t > 0.$$

A voltage equal to zero is maintained at $x = l$, while at the end $x = 0$, the voltage varies according to the law

$$v(0,t) = Ct, \qquad t > 0,$$

where C is a constant. Find $v(x,t)$ if the initial voltage distribution is zero.

19. Establish the validity of the formal solution of the initial boundary-value problem:

$$\begin{aligned} u_t &= k\, u_{xx}, && 0 < x < l, && t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, && \\ u(0,t) &= 0, && u_x(l,t) = 0, && t \ge 0. \end{aligned}$$

20. Prove the uniqueness of the solution of the problem:

$$\begin{aligned} u_t &= k\, u_{xx}, && 0 < x < l, && t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, && \\ u_x(0,t) &= 0, && u_x(l,t) = 0, && t \ge 0. \end{aligned}$$

21. Solve the radioactive decay problem:

$$\begin{aligned} u_t - k\,u_{xx} &= Ae^{-ax}, && 0 < x < \pi, && t > 0, \\ u(x,0) &= \sin x, && 0 \le x \le \pi, \\ u(0,t) &= 0, && u(\pi,t) = 0, && t \ge 0. \end{aligned}$$

22. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_t - k\,u_{xx} &= h(x,t), && 0 < x < l, && t > 0, \quad k = \text{constant}, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u(0,t) &= p(t), && u(l,t) = q(t), && t \ge 0. \end{aligned}$$

23. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_t - k\,u_{xx} &= h(x,t), && 0 < x < l, && t > 0, \\ u(x,0) &= f(x), && 0 \le x \le l, \\ u(0,t) &= p(t), && u_x(l,t) = q(t), && t \ge 0. \end{aligned}$$

24. Solve the problem:

$$\begin{aligned} u_t - k\,u_{xx} &= 0, && 0 < x < 1, && t > 0, \\ u(x,0) &= x(1-x), && 0 \le x \le 1, \\ u(0,t) &= t, && u(1,t) = \sin t, && t \ge 0. \end{aligned}$$

25. Solve the problem:

$$\begin{aligned} u_t - 4u_{xx} &= xt, && 0 < x < 1, && t \ge 0, \\ u(x,0) &= \sin \pi x, && 0 \le x \le 1, \\ u(0,t) &= t, && u(1,t) = t^2, && t \ge 0. \end{aligned}$$

26. Solve the problem:

$$\begin{aligned} u_t - k\,u_{xx} &= x\cos t, && 0 < x < \pi, && t > 0, \\ u(x,0) &= \sin x, && 0 \le x \le \pi, \\ u(0,t) &= t^2, && u(\pi,t) = 2t, && t \ge 0. \end{aligned}$$

27. Solve the problem:

$$\begin{aligned} u_t - u_{xx} &= 2x^2 t, && 0 < x < 1, && t > 0, \\ u(x,0) &= \cos(3\pi x/2), && 0 \le x \le 1, \\ u(0,t) &= 1, && u_x(1,t) = \frac{3\pi}{2}, && t \ge 0. \end{aligned}$$

28. Solve the problem:

$$\begin{aligned} u_t - 2\,u_{xx} &= h, && 0 < x < 1, && t > 0, \quad h = \text{constant}, \\ u(x,0) &= x, && 0 \le x \le 1, \\ u(0,t) &= \sin t, && u_x(1,t) + u(1,t) = 2, && t \ge 0. \end{aligned}$$

29. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le l, \\ u_t(x,0) &= g(x), \quad 0 \le x \le l, \\ u(0,t) &= p(t), \quad u_x(l,t) = q(t), \quad t \ge 0. \end{aligned}$$

30. Determine the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= h(x,t), \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le l, \\ u_t(x,0) &= g(x), \quad 0 \le x \le l, \\ u_x(0,t) &= p(t), \quad u_x(l,t) = q(t), \quad t \ge 0. \end{aligned}$$

31. Solve the problem:

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= x, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) &= t^2, \quad u(1,t) = \cos t, \quad t \ge 0. \end{aligned}$$

32. Solve the problem:

$$\begin{aligned} u_{tt} - 4\,u_{xx} &= xt, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= x, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad u_x(1,t) = 1 + t, \quad t \ge 0. \end{aligned}$$

33. Solve the problem:

$$\begin{aligned} u_{tt} - 9\,u_{xx} &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= \sin\left(\frac{\pi x}{2}\right), \quad u_t(x,0) = 1 + x, \quad 0 \le x \le 1, \\ u_x(0,t) &= \pi/2, \quad u_x(1,t) = 0, \quad t \ge 0. \end{aligned}$$

34. Find the solution of the problem:

$$\begin{aligned} u_{tt} + 2k\,u_t - c^2 u_{xx} &= 0, \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 \le x \le l, \\ u_x(0,t) &= 0, \quad u(l,t) = h, \quad t \ge 0, \quad h = \text{constant}. \end{aligned}$$

35. Solve the problem:

$$\begin{aligned} u_t - c^2 u_{xx} + hu &= hu_0, \quad -\pi < x < \pi, \quad t > 0, \\ u(x,0) &= f(x), \quad -\pi \le x \le \pi, \\ u(-\pi,t) &= u(\pi,t), \quad u_x(-\pi,t) = u_x(\pi,t), \quad t \ge 0, \end{aligned}$$

where h and u_0 are constants.

36. Prove the uniqueness theorem for the boundary-value problem involving the Laplace equation:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < a, && 0 < y < b, \\ u(x,0) &= f(x), \quad u(x,b) = 0, & && 0 \le x \le a, \\ u_x(0,y) &= 0 = u_x(a,y), & && 0 \le y \le b. \end{aligned}$$

37. Consider the telegraph equation problem:

$$\begin{aligned} u_{tt} - c^2 u_{xx} + a u_t + b u = 0, \qquad & 0 < x < l, \qquad t > 0, \\ u(x,0) = f(x), \qquad u_t(x,0) = g(x) \quad & \text{for} \quad 0 \le x \le l, \\ u(0,t) = 0 = u(l,t) \quad & \text{for} \quad t \ge 0, \end{aligned}$$

where a and b are positive constants.

(a) Show that, for any $T > 0$,

$$\int_0^l \left(u_t^2 + c^2 u_x^2 + b u^2\right)_{t=T} dx \le \int_0^l \left(u_t^2 + c^2 u_x^2 + b u^2\right)_{t=0} dx.$$

(b) Use the above integral inequality from (a) to show that the initial boundary-value problem for the telegraph equation can have only one solution.