DISCRETE STRUCTURE LECTURE NOTES-2

BSc.(H) Computer Science: II Semester

Teacher: Ms. Sonal Linda

PLANAR GRAPHS

What is Planar Graph?

A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such drawing is called a planar representation of the graph.

Example 1: Is K_4 planar (shown in figure 1 with two edges crossing)?

Solution: K_4 is planar because it can be drawn without crossings as shown in Figure 2.

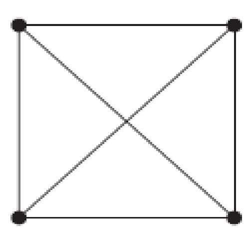


FIGURE 1 The Graph K_4.

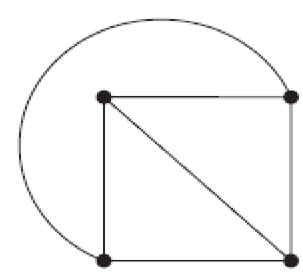


FIGURE 2 K_4 Drawn with No Crossings.

Planar Graphs

Example 2: Is Q_3 shown in Figure 3 planar?

Solution: Q_3 is planar because it can be drawn without crossings as shown in Figure 4.

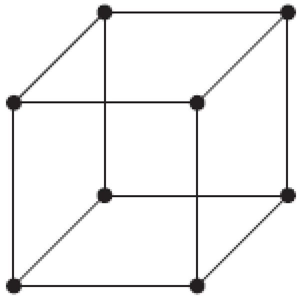


FIGURE 3 The Graph Q_3.

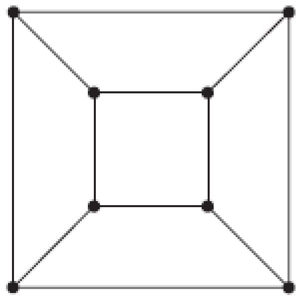


FIGURE 4 A Planar Representation of Q_3.

Planar Graphs

Example 3: Is $K_{3,3}$ shown in Figure 5 planar?

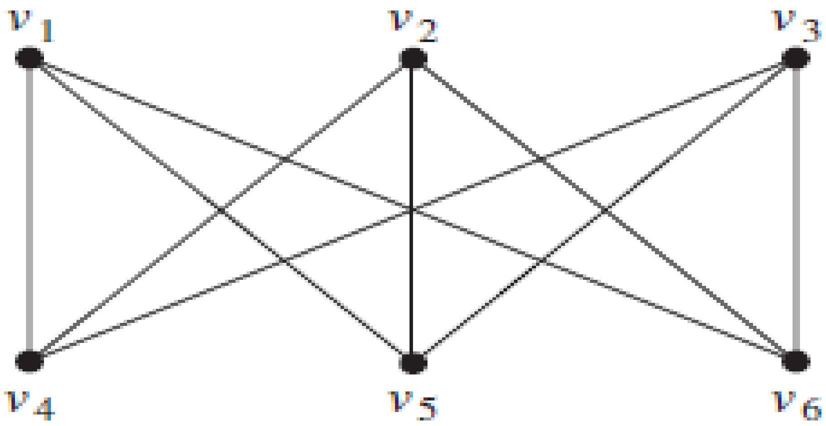


FIGURE 5 **The Graph $K_{3,3}$.**

Planar Graphs

Solution: $K_{3,3}$ is not planar because it can not be drawn without crossings.

- In any planar representation of $K_{3,3}$, the vertices $v_1$and v_2 must be connected to both $v_4$and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2 shown in Figure 6(a).

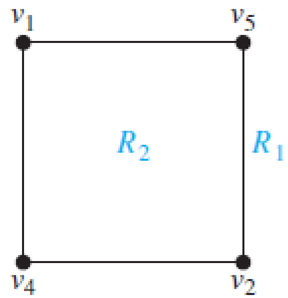


FIGURE 6 (a) Showing that $K_{3,3}$ Is Nonplanar.

Planar Graphs

Solution:

- The vertex v_3 is in either R_1or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between $v_3$and v_4 and between $v_3$and v_5 separate R_2 into sub regions, R_{21} and R_{22} as shown in Figure 6(b).

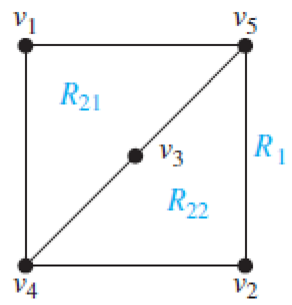


FIGURE 6 (b) Showing that $K_{3,3}$ Is Nonplanar.

Euler’s Formula

Let G be a connected planner simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $\boldsymbol{r = e - v + 2}$

A planar representation of G is shown in Figure 7.

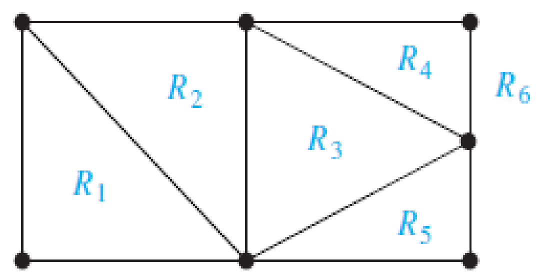


FIGURE 7 The Regions of the Planar Representation of a Graph.

Euler's Formula

Example 1: Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a regions does a representation of this planar graph split the plane?

Solution: This graph has 20 vertices, each of degree of 3, so $v = 20$. Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60\ or\ e = 30$. Consequently, from Euler's formula, the number of regions is:

$$\boldsymbol{r = e - v + 2}$$

$$\boldsymbol{r = 30 - 20 + 2 = 12}$$

Euler's Formula

Corollary 1: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.

Corollary 2 : If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Example 1: Show that K_5 is non-planar using corollary 1.

Solution: The graph K_5 has 5 vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$ and $3v - 6 = 9$. Therefore, K_5 is non-planar.

Euler's Formula

Corollary 3: If a connected planar simple graph has e edges and v vertices with $v \geq 3$, and no circuits of length of 3, then $e \leq 2v - 4$.

Example 1: Show that $K_{3,3}$ is non-planar using Corollary 3.

Solution: Because $K_{3,3}$ has no circuits of length 3 (this is easy to see because it is bipartite), Corollary 3 can be used. $K_{3,3}$ has 6 vertices and 9 edges. Because, $e = 9$ and $2v - 4 = 8$, Corollary 3 shows that $K_{3,3}$ is non-planar.

Kuratowski's Theorem

- If a graph is *planar*, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*.
- The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homeomorphic* if they can be obtained from the same graph by a *sequence* of *elementary subdivision*.

Theorem: A graph is non-planar if and only if it contains a subgraph *homeomorphic* to $K_{3,3}$ or K_5.

Kuratowski's Theorem

Example 1: Show that the graphs G_1, G_2 and G_3 in Figure 8 are all homeomorphic.

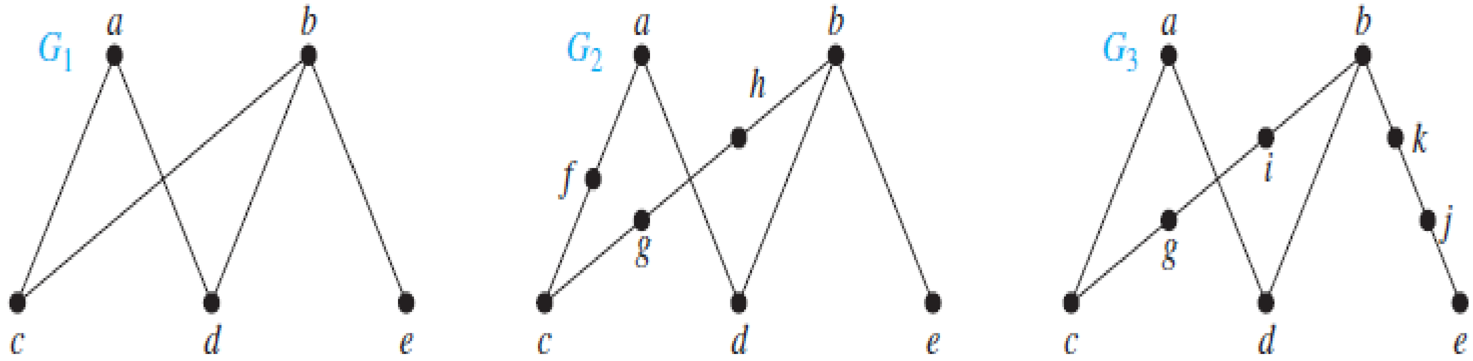


FIGURE 8 Homeomorphic Graphs.

Kuratowski's Theorem

Solution: These three graphs are homeomorphic because all three can be obtained from G_1 by elementary subdivisions.

- G_1 can be obtained from itself by an empty sequence of elementary subdivisions.
- To obtain G_2 from G_1 we can use this sequence of elementary subdivisions:
  - Remove the edge $\{a, c\}$, add the vertex f, and add the edge $\{a, f\}$ and $\{f, c\}$.
  - Remove the edge $\{b, c\}$, add the vertex g, and add the edge $\{b, g\}$ and $\{g, c\}$.
  - Remove the edge $\{b, g\}$, add the vertex h, and add the edge $\{g, h\}$ and $\{b, h\}$.

- Do the same for G_3.

Kuratowski’s Theorem

Example 2: Determine whether the graph G shown in Figure 9 is planar.

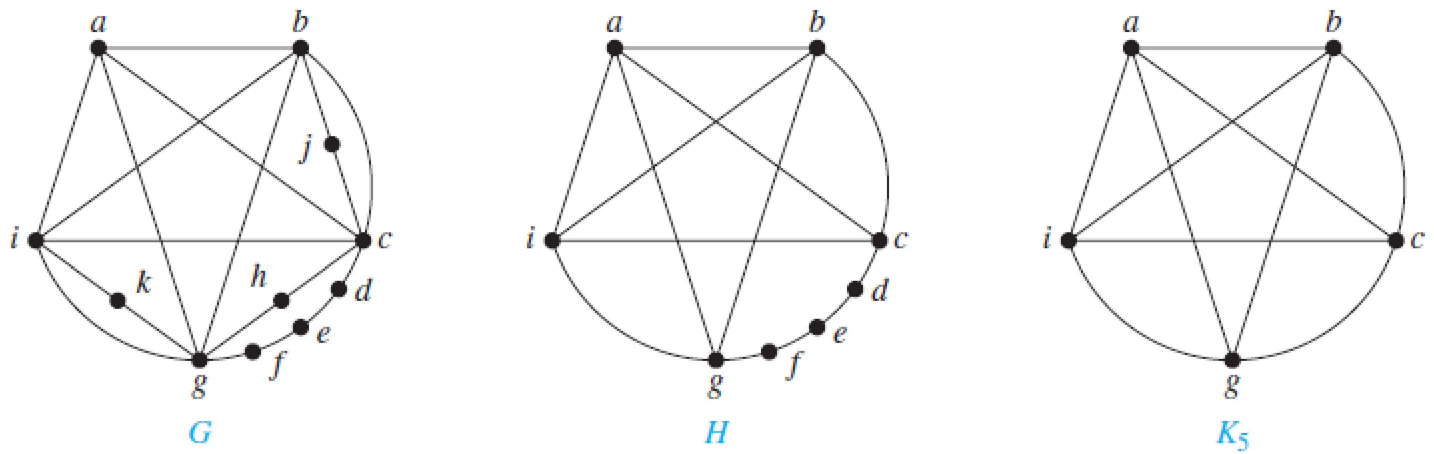


FIGURE 9 The Undirected Graph G, a Subgraph H Homeomorphic to K_5, and K_5.

Kuratowski's Theorem

Solution: The graph G has a subgraph H homeomorphic to K_5.

- H is obtained by deleting h, j and k and all edges incident with these vertices.
- H is homeomorphic to K_5 because it can be obtained from K_5 (with vertices a, b, c, g and i) by a sequence of elementary subdivisions, adding the vertices d, e, and f.
- Hence, G is non-planar.

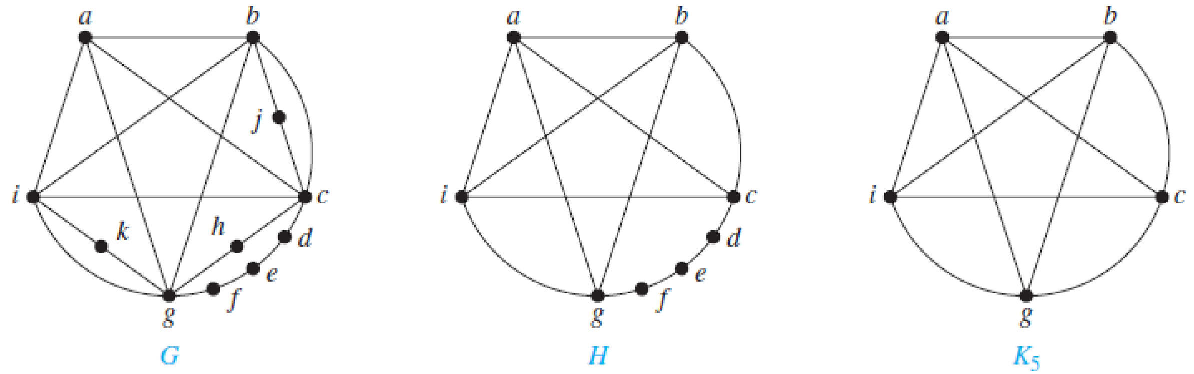
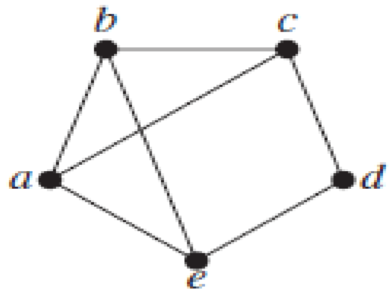


FIGURE 9 The Undirected Graph G, a Subgraph H Homeomorphic to K_5, and K_5.

Exercises : Euler and Hamilton Paths

In Exercises 1–8 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1.



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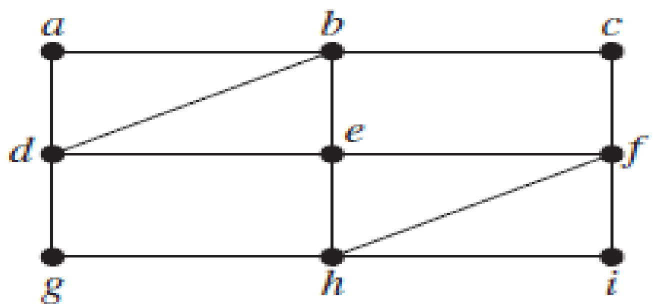

Exercises: Euler and Hamilton Paths

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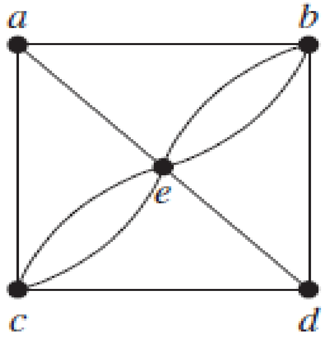

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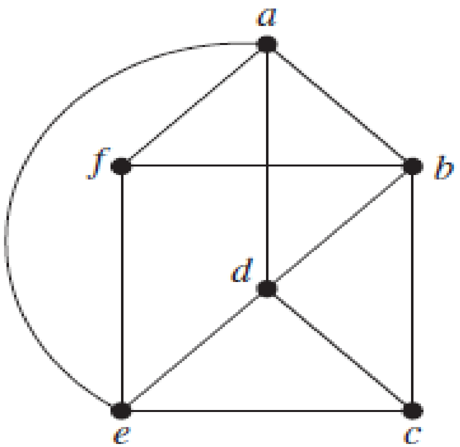

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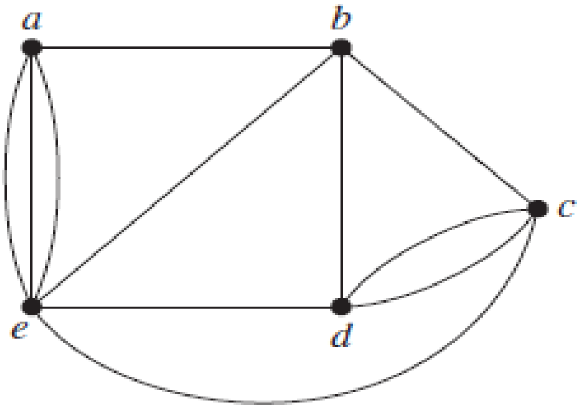

Exercises: Euler and Hamilton Paths

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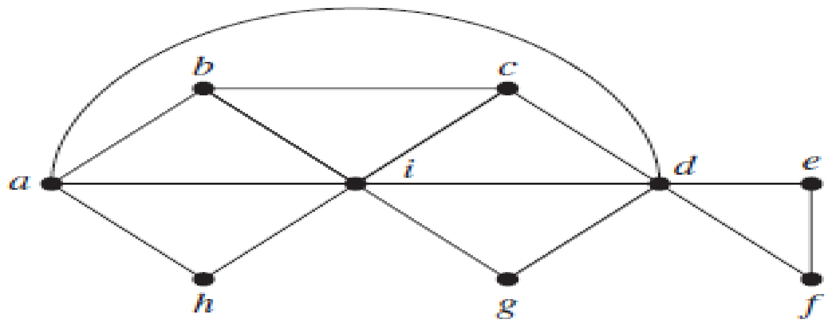

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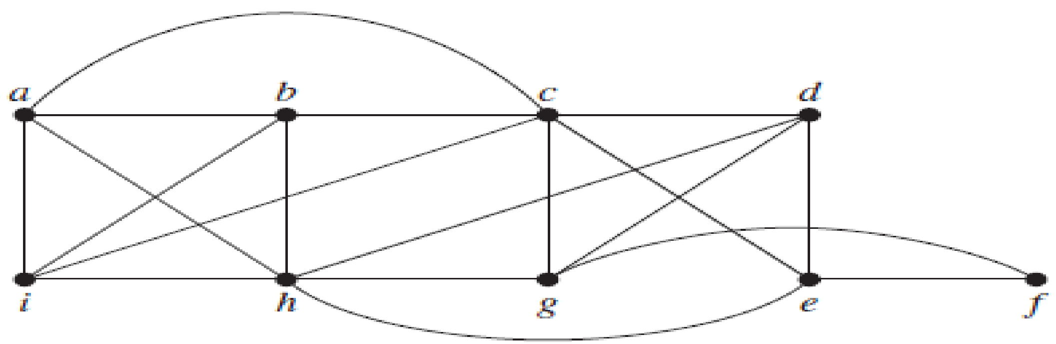

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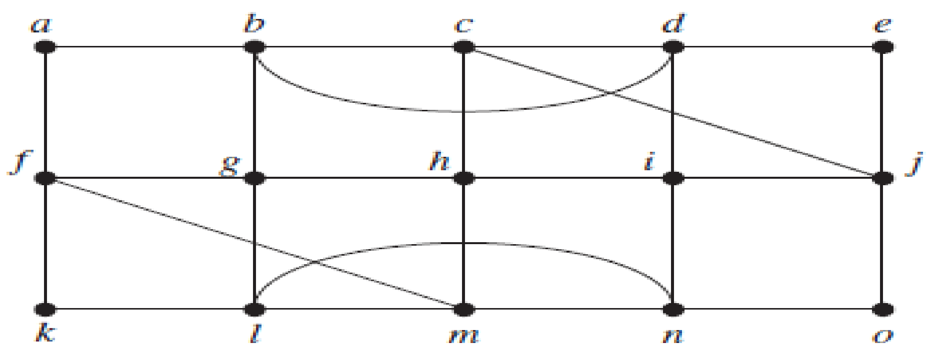
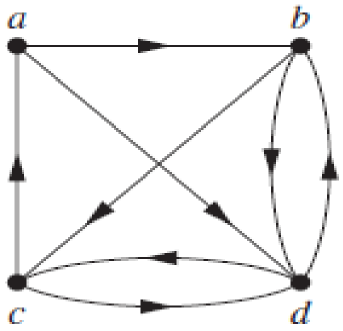


Exercises: Euler and Hamilton Paths

In Exercises 18–23 determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.

18.



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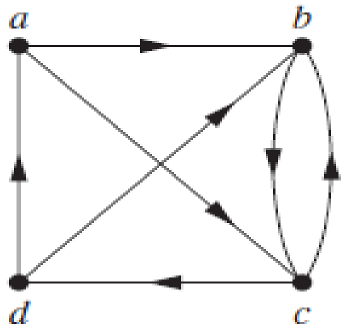

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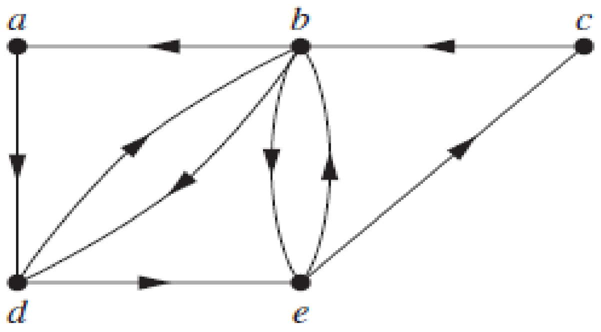

Exercises: Euler and Hamilton Paths

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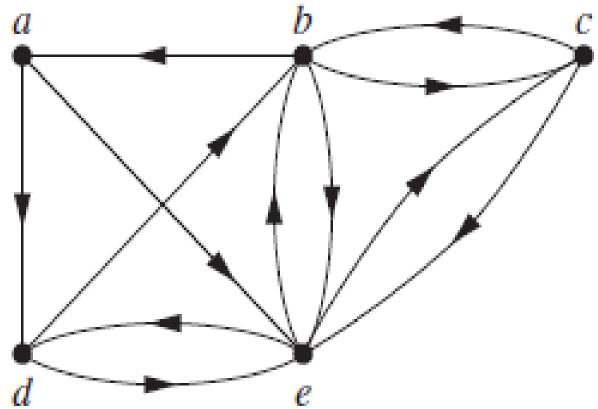

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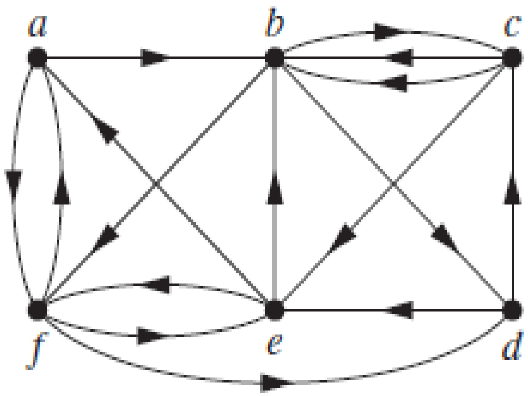

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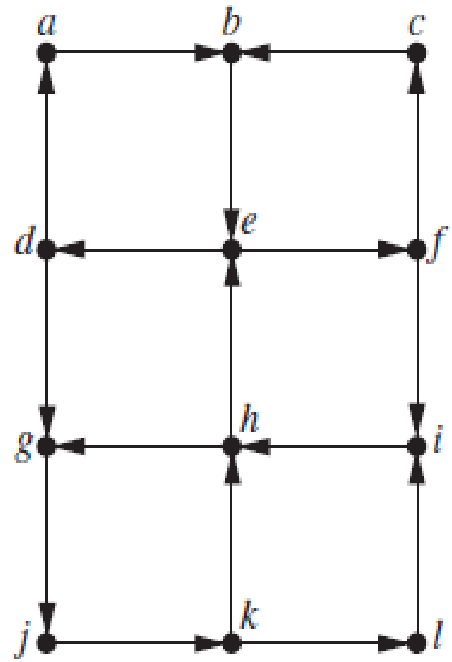
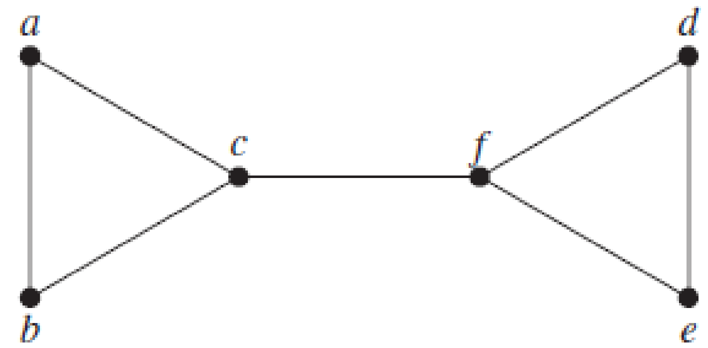


Exercises: Euler and Hamilton Paths

In Exercises 30–36 determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

30.



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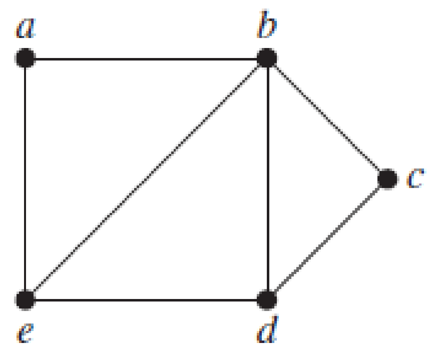

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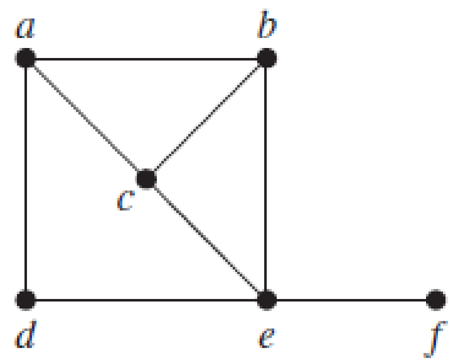

Exercises: Euler and Hamilton Paths

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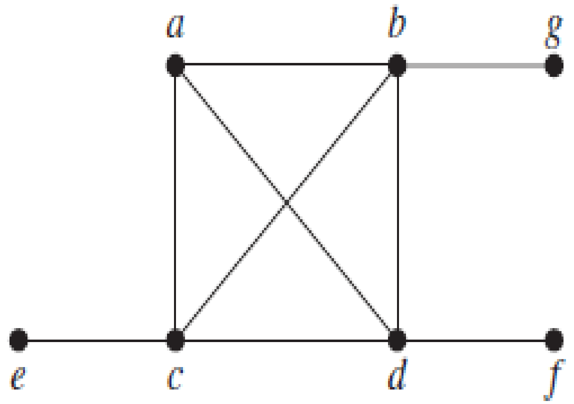

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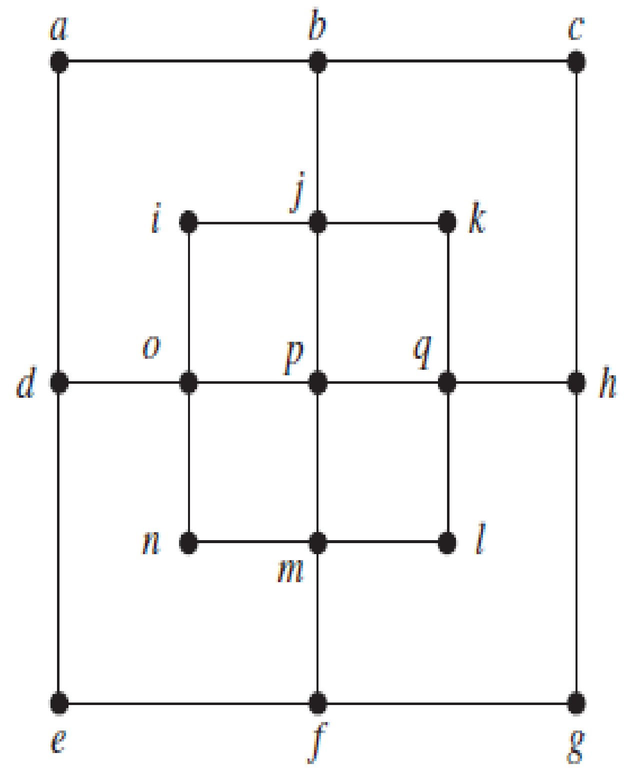

Exercises: Euler and Hamilton Paths

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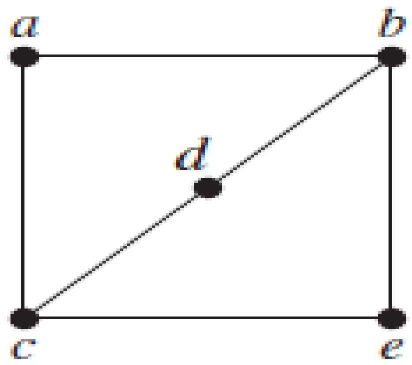

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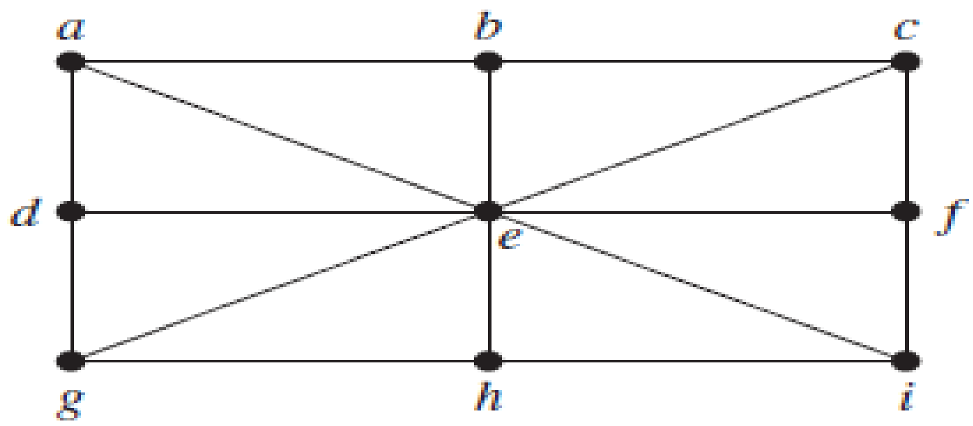
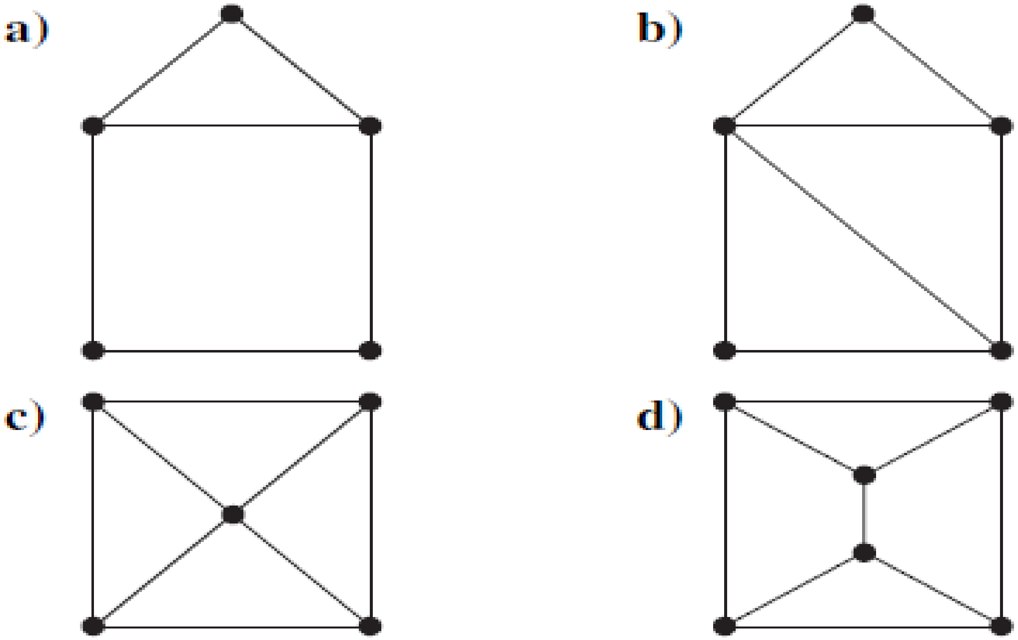


Exercises: Euler and Hamilton Paths

47. For each of these graphs, determine (*i*) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit, (*ii*) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (*iii*) whether the graph has a Hamilton circuit.



Exercises: Planar Graphs

In Exercises 2–4 draw the given planar graph without any crossings.

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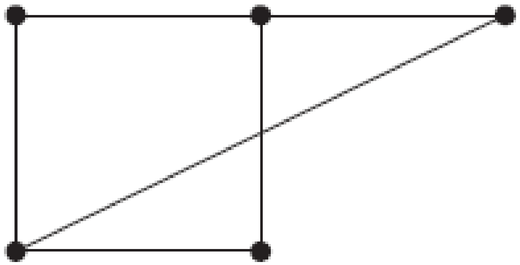

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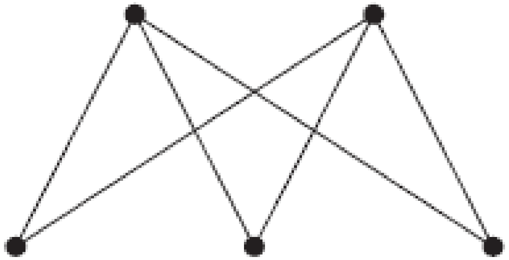

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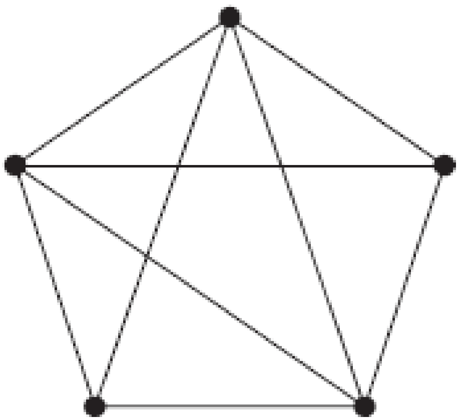
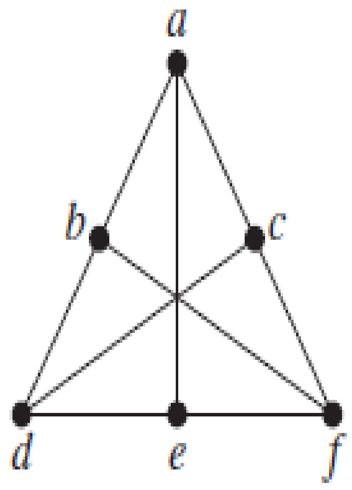


Exercises: Planar Graphs

In Exercises 5–9 determine whether the given graph is planar. If so, draw it so that no edges cross.

5.



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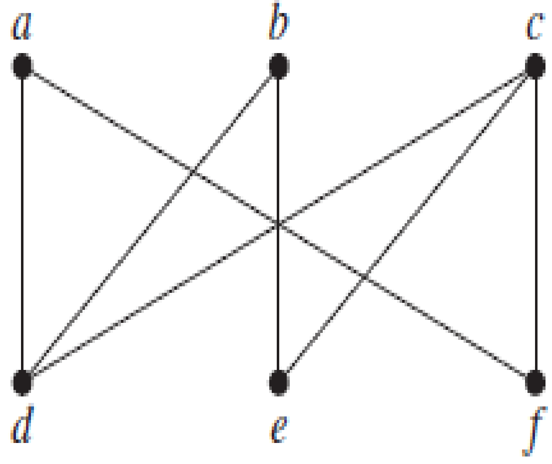

Exercises: Planar Graphs

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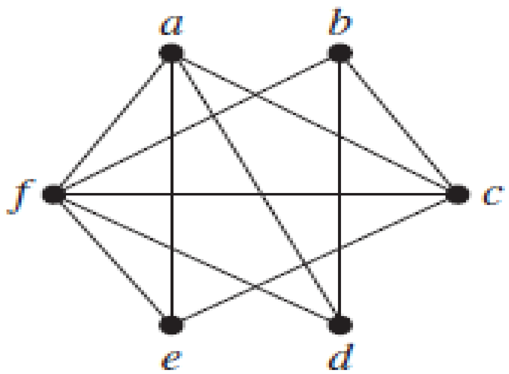

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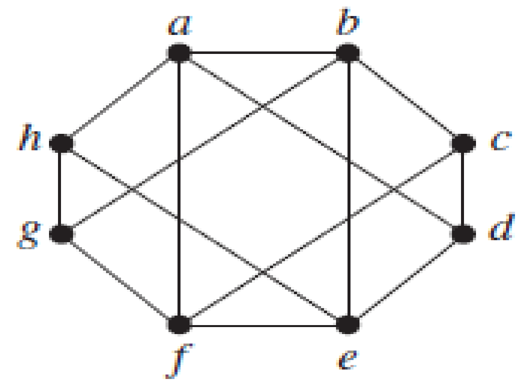

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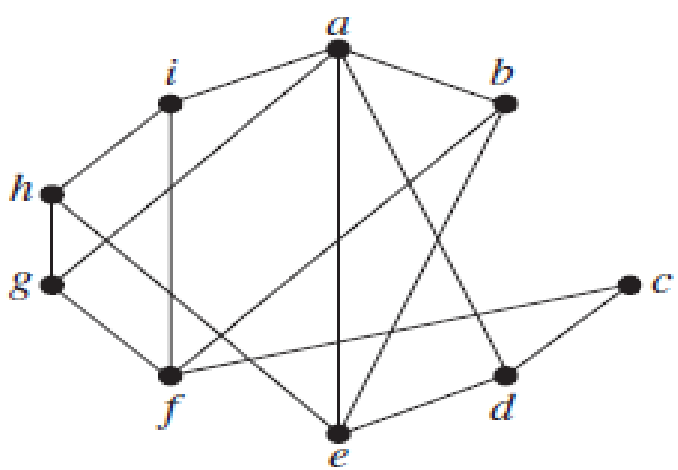
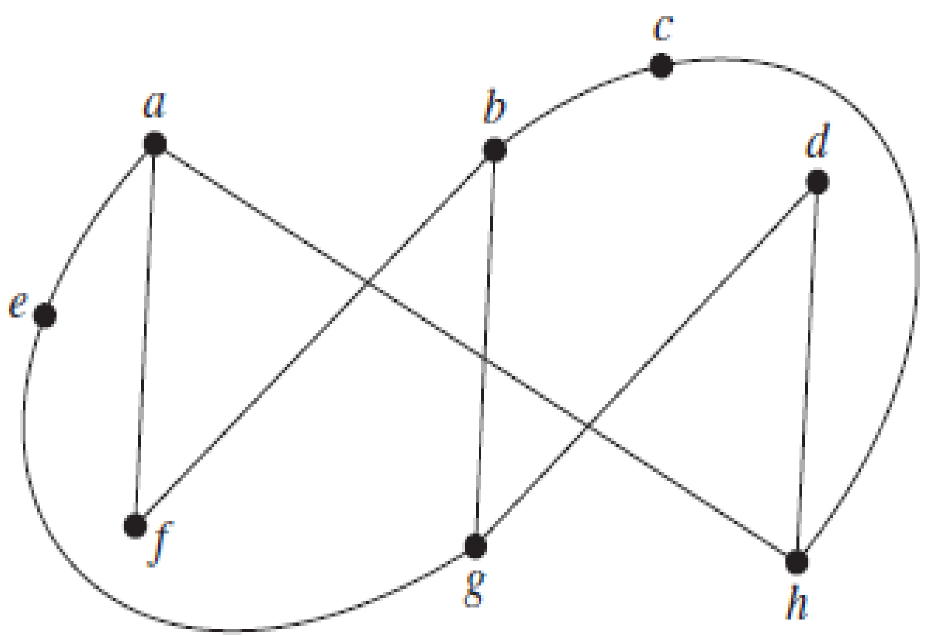


Exercises: Planar Graphs

In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

20.



Exercises: Planar Graphs

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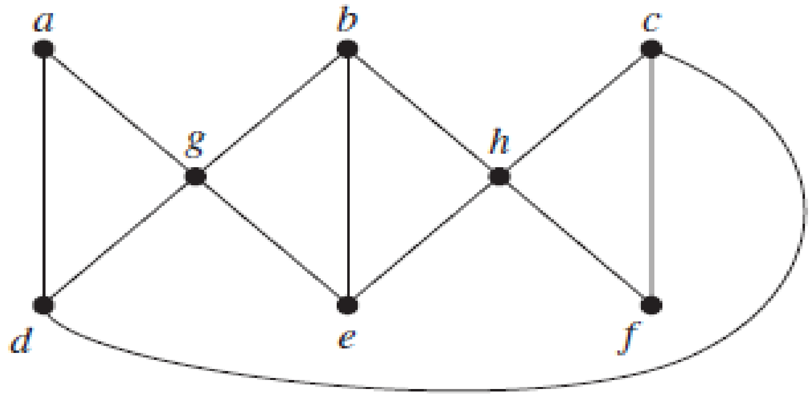

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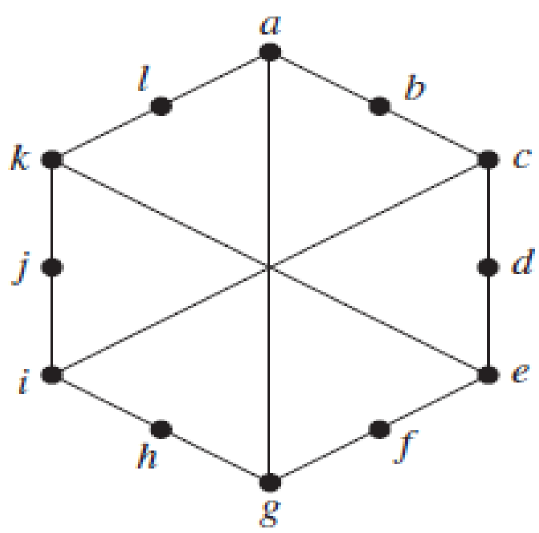

Exercises: Planar Graphs

In Exercises 23–25 use Kuratowski's theorem to determine whether the given graph is planar.

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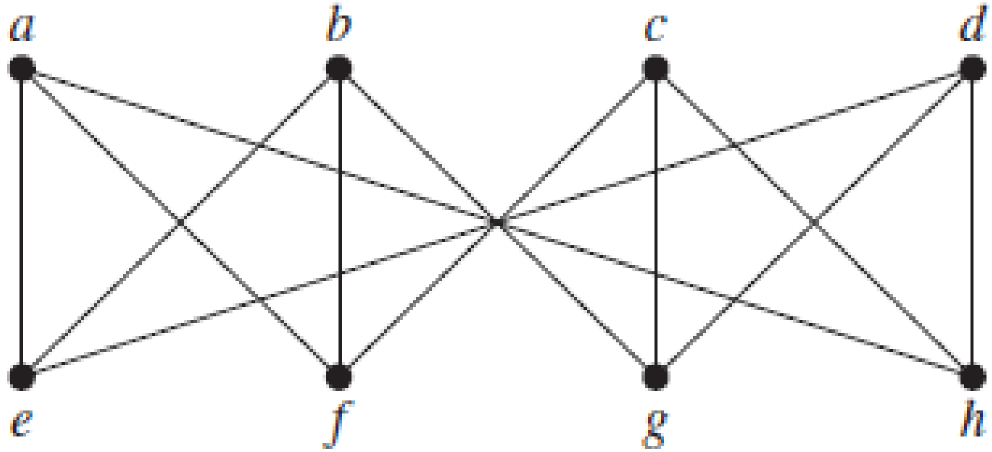

Exercises: Planar Graphs

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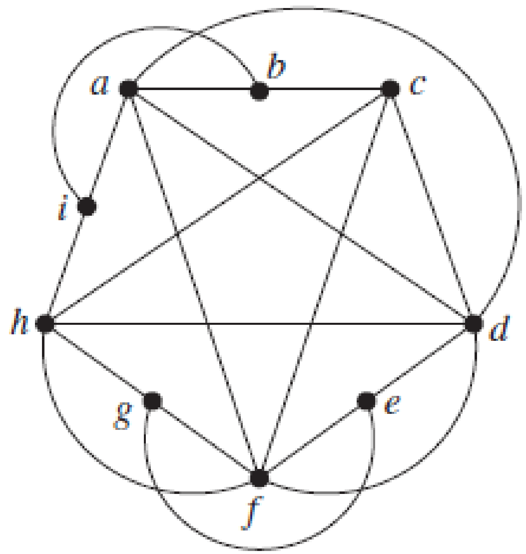

25.