Second order PDE, homogeneous wave equation, initial boundary value problems, non-homogeneous boundary conditions, finite strings with fixed ends, non-homogeneous wave equation, Riemann problem, Goursat problem

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We have discussed Second order PDE, homogeneous wave equation, initial boundary value problem up-to the Semi-infinite String with a Fixed End in the classroom:

1 Second order PDE, homogeneous wave equation

1.1 Semi-infinite vibrating String with a Free End

The model of the problem can be written as;
a semi-infinite string with a free end at $x = 0$.

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \tag{1}$$

with the initial conditions

$$u(x, 0) = f(x), \quad 0 \le x < \infty$$

$$u_t(x,0) = g(x), \quad 0 \le x < \infty,$$

$$u_x(0,t) = 0, 0 \le t < \infty \quad \text{i.e. the gradient of the distance at} x = 0 \quad \text{is} \quad 0$$

We know that the equation (1) is of the hyperbolic form. General solution of (1) is

$$u(x,t) = \phi(x+ct) + \psi(x-ct) \tag{2}$$

As in the case of the fixed end, for $x > ct$ the solution is the same as that of the infinite string.

$$u(x,0) = f(x) = \phi(x) + \psi(x) \tag{3}$$

Now differentiating partially (2) w.r.t. "t" we have

$$u_t(x,t) = \phi'(x+ct)\frac{\partial}{\partial t}(x+ct) + \psi'(x-ct)\frac{\partial}{\partial t}(x-ct)$$

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$$

Now integrating the equation $c\phi'(x) - c\psi'(x) = g(x)$ we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_0^x g(\tau)\,d\tau + K \tag{4}$$

From (3) & (4) we have

$$\phi(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_0^x g(\tau)\,d\tau + K\right] \tag{5}$$

and

$$\psi(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_0^x g(\tau)\,d\tau - K\right] \tag{6}$$

So that the general solution for $x > ct$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)\,d\tau - \int_0^{x-ct} g(\tau)\,d\tau\right]$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{x+ct} g(\tau)\,d\tau\right] \tag{7}$$

Now for $x < ct$, we differentiating partially (2) w.r.t. "x" we have

$$u_x(x,t) = \phi'(x+ct)\frac{\partial}{\partial x}(x+ct) + \psi'(x-ct)\frac{\partial}{\partial x}(x-ct)$$

$$u_x(x,t) = \phi'(x+ct)\psi'(x-ct)$$
$$u_x(0,t) = \phi'(ct) + \psi'(-ct) = 0$$

Now integrating the equation $\phi'(ct) + \psi'(-ct) = 0$ we have

$$\phi(ct) - \psi(-ct) = K \tag{8}$$

where M is a constant. If we let $\alpha = -ct$, we obtain

$$\psi(\alpha) = \phi(-\alpha) - K$$

Now replace $\alpha = x - ct$, we have,

$$\psi(x-ct) = \phi(ct-x) - K$$

$$\psi(x-ct) = \frac{1}{2}\left[f(ct-x) + \frac{1}{c}\int_0^{ct-x} g(\tau)\,d\tau\right] - K/2$$

so the solution be

$$u(x,ct) = \phi(x+ct) + \psi(x-ct)$$
$$u(x,ct) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)\,d\tau + \int_0^{ct-x} g(\tau)\,d\tau\right]$$

Note Also see another pdf file.