Interpolation: Lagrange's form and Newton's form Finite difference operators, Gregory Newton forward and backward differences Interpolation.

Dr. Chandrashekhar Nishad
Department of Mathematics,
Aryabhatta College,
University of Delhi.
Email: shekhar.nish@gmail.com

We have discussed about Lagrange's form and Newton's forms Interpolations in the class which was held previously however:

1 NEWTON INTERPOLATING POLYNOMIAL

1.1 Linear

$$P(x) = a_0 + (x - x_0)a_1 \tag{1}$$

where a_0 and a_1 are arbitrary constants, which satisfy the conditions $f(x_0) = P(x_0)$ & $f(x_1) = P(x_1)$. We have

$$P(x_0) = f(x_0) = a_0 + (x_0 - x_0)a_1 \Rightarrow f(x_0) = a_0$$

and

$$P(x_1) = f(x_1) = f(x_0) + (x_1 - x_0)a_1 \Rightarrow \frac{f(x_1) - f(x_0)}{x_1 - x_0} = a_1$$

So by eliminating a_0 and a_1 from equation (1), the Newton's linear-interpolation formula:

$$P(x) = f(x_0) + (x - x_0)f[x_0, x_1] \tag{2}$$

where

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

1.2 Quadratic Interpolation

$$P(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 \tag{3}$$

where a_0, a_1 & a_3 are arbitrary constants, which satisfy the conditions $f(x_0) = P(x_0), f(x_1) = P(x_1) \quad \& \quad f(x_2) = P(x_2)$. We have

$$P(x_0) = f(x_0) = a_0 + (x_0 - x_0)a_1 + (x_0 - x_0)(x_0 - x_1)a_2 \Rightarrow f(x_0) = a_0$$

$$P(x_1) = f(x_1) = f(x_0) + (x_1 - x_0)a_1 + (x_1 - x_0)(x_1 - x_1)a_2$$

$$\Rightarrow \frac{f(x_1) - f(x_0)}{x_1 - x_0} = a_1 = f[x_0, x_1]$$

and

$$P(x_2) = f(x_2) = f(x_0) + (x_1 - x_0)f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1)a_2$$

$$\Rightarrow \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0} = a_2 = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = f[x_0, x_1, x_2]$$

So by eliminating a_0, a_1 and a_2 from equation (3), the Newton's Quadratic-interpolation formula:

$$P(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x -_1)f[x_0, x_1, x_2] \tag{4}$$

1.3 General Form of Newton's Interpolating Polynomials

$$P(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + \cdots + (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})a_n \tag{5}$$

where $a_i = f[x_0, x_1, \cdots, x_i]$ and $f[x_0, x_1, \cdots, x_i] = \frac{f[x_1, x_2, \cdots, x_i] - f[x_0, x_1, \cdots, x_{i-1}]}{x_i - x_0}$
Also construct Newton's divided difference table.

Table 1: Divided Difference Table

		1st divided difference	2nd divided difference	3rd divided difference
x_0	$f[x_0]$			
x_1	$f[x_1]$	$f[x_0, x_1]$		
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_{1,2}]$	
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_{2,3}]$	$f[x_0, x_1, x_2, x_3]$

Note: Kindly do some examples.

2 LAGRANGE INTERPOLATING POLYNOMIAL

2.1 Linear

A linear polynomial

$$P(x) = a_0 + a_1 x$$

where a_0 and a_1 are arbitrary constants, which satisfy the conditions $f(x_0) = P(x_0) \quad \& \quad f(x_1) = P(x_1)$. So by eliminating a_0 and a_1 from above equation we have,

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) \tag{6}$$

where $L_0(x) = \frac{x-x_1}{x_0-x_1} \quad \& \quad L_1(x) = \frac{x-x_0}{x_1-x_0}$

2.2 Quadratic

Similarly
A linear polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2$$

where a_0, a_1 and a_2 are arbitrary constants, which satisfy the conditions $f(x_0) = P(x_0), \quad f(x_1) = P(x_1) \quad \& \quad f(x_2) = P(x_2)$. So by eliminating a_0, a_1 and a_2 from above equation we have,

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) \tag{7}$$

where $L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \quad \& \quad L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

2.3 General Form of Lagrange's Interpolating Polynomials

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + \cdots + L_n(x)f(x_n) \qquad (8)$$

where

$$L_i(x)\frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Note: Kindly do some examples.

3 FINITE DIFFERENCE OPERATORS

Let there be a closed interval $[a, b]$ and it is divided into n equal sub-intervals such as; $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Let h be the length of each sub-interval, so $h = \frac{b-a}{n}$

Now $a = x_0,\ x_1 = a + h = x_0 + h = x_1,\ x_2 = x_0 + 2h = x_2,\ \cdots,\ x_i = x_0 + ih = x_{i+1},\ \cdots\ x_n = x_0 + nh = b$

Definitions:

Shift Operator

$$Ef(x_i) = f(x_i + h) = f(x_{i+1})$$

Forward Difference Operator

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$$

Backward Difference Operator

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f(x_i) - f(x_{i-1})$$

Central Difference Operator

$$\delta f(x_i) = f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2})$$

Average Operator

$$\mu f(x_i) = \frac{1}{2}\left[f(x_i + \frac{h}{2}) + f(x_i - \frac{h}{2})\right]$$

Relationships between operators:

$$E^m f(x_i) = f(x_i + mh)$$

$$\Delta f(x_i) = f(x_i+h)-f(x_i) = Ef(x_i)-f(x_i) = (E-1)f(x_i)$$

$$\Rightarrow \Delta = E - 1;$$

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f(x_i) - E^{-1}f(x_i)$$

$$\Rightarrow \nabla = 1 - E^{-1};$$

$$\delta f(x_i) = f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2}) = E^{\frac{1}{2}}f(x_i) - E^{-\frac{1}{2}}f(x_i)$$

$$\Rightarrow \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}};$$

and

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

Note: • Kindly read reference book [1] page no. 230, 231, 232,233, 234, 235, 236 & 237. Also do the exercise question based on example 4.12

• Kindly solve the example 4.15 and at least one exercise question based on example 4.15 using Gregory-Newton Forward (and backward) Difference Interpolation.

• I will attached the pics of these pages. Any doubt you people ask by phone or whats app or by email. Rest of reading material will be send soon. Thanks